

# BAR MODE INSTABILITY IN RELATIVISTIC ROTATING STARS: A POST-NEWTONIAN TREATMENT

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## ABSTRACT

We construct analytic models of incompressible, uniformly rotating stars in post-Newtonian (PN) gravity and evaluate their stability against nonaxisymmetric bar modes. We model the PN configurations by homogeneous triaxial ellipsoids and employ an energy variational principle to determine their equilibrium shape and stability. The spacetime metric is obtained by solving Einstein’s equations of general relativity in 3+1 ADM form. We use an approximate subset of these equations well-suited to numerical integration in the case of strong field, three dimensional configurations in quasi-equilibrium. However, the adopted equations are exact at PN order, where they admit an analytic solution for homogeneous ellipsoids. We obtain this solution for the metric, as well as analytic functionals for the conserved global quantities,  $M$ ,  $M_0$  and  $J$ .

We present sequences of axisymmetric, rotating equilibria of constant density and rest mass parametrized by their eccentricity. These configurations represent the PN generalization of Newtonian Maclaurin spheroids, which we compare to other PN and full relativistic incompressible equilibrium sequences constructed by previous investigators. We employ the variational principle to consider nonaxisymmetric ellipsoidal deformations of the configurations, holding the angular momentum constant and the rotation uniform. We locate the point along each sequence at which these Jacobi-like bar modes will be driven secularly unstable by the presence of a dissipative agent like viscosity. We find that the value of the eccentricity, as well as related ratios like  $\Omega^2/(\pi\rho_0)$  and  $T/|W|$  (= rotational kinetic energy / gravitational potential energy), defined invariantly, all increase at the onset of instability as the stars become more relativistic. Since higher degrees of rotation are required to trigger a viscosity-driven bar mode instability as the stars become more compact, the effect of general relativity is to weaken the instability, at least to PN order. This behavior is in stark contrast to that found recently for secular instability via nonaxisymmetric, Dedekind-like modes driven by gravitational radiation. These findings support the suggestion that in general relativity nonaxisymmetric modes driven unstable by viscosity no longer coincide with those driven unstable by gravitational radiation.

*Subject headings:* gravitation — relativity — instabilities — stars: neutron — stars: rotation

## 1. Introduction

The identification of nonaxisymmetric modes of instability in rapidly rotating equilibrium configurations is a classic problem. Although a considerable amount of work has been done in Newtonian theory, where

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a number of results are well established (see Chandrasekhar 1969a, hereafter Ch69 and section 2 for a summary and references), only a few investigations have been carried out so far in the context of general relativity. The first three dimensional (3D) perturbation computations in full general relativity to identify nonaxisymmetric instabilities driven by gravitational radiation have been carried out recently by Stergioulas & Friedman (1997, hereafter SF) and Stergioulas (1997). A earlier numerical investigation of the effects of relativity on the viscosity-driven bar mode instability was presented by Bonazzola, Friebe & Gourgoulhon (1996, hereafter BFG) and these results have been recently corroborated by a more detailed analysis (Bonazzola, Friebe & Gourgoulhon 1997). SF solved the coupled set of perturbed field equations, modeling uniformly rotating stars by a polytropic equation of state (EOS) and adopting the Friedman & Schutz (1995) criterion for the onset of the nonaxisymmetric instability to gravitational radiation dissipation. According to this criterion, a nonaxisymmetric mode becomes unstable when its frequency, as measured by an observer at infinity, vanishes. They find that relativistic models are unstable to nonaxisymmetric modes for significantly smaller degrees of rotation than for corresponding Newtonian models. The destabilizing effect of relativity is most striking in the case of the  $m = 2$  mode, which can become unstable even for soft polytropes of index  $\gamma \geq \gamma_{crit} = 1.77$  ( $n \leq 1.3$ ), while the critical index in Newtonian theory is  $\gamma_{crit} = 2.238$  (Jeans 1919, 1928; James 1964). This behaviour is in agreement with results of a semianalytical, PN analysis previously presented by Cutler (1991) and Cutler & Lindblom (1992, see also Lindblom 1995, Yoshida & Eriguchi 1997). In particular, Cutler (1991) derived the PN corrections to the pulsational modes of uniformly rotating stars. The resulting expressions were then used by Cutler & Lindblom (1992) to evaluate the critical value of the star angular velocity,  $\Omega_{crit}$ , where the frequency of the mode passes through zero. This is the critical value at which, in absence of viscosity, these modes are unstable to the emission of gravitational radiation. By considering the  $\gamma = 2$  polytrope, they found that PN effects lowers by up to 10 % the values of  $\Omega_{crit}$ , concluding that PN effects tend to make the gravitational-radiation instability more important. However, as noted by SF, the results of BFG, who investigated the effects of general relativity on the  $m = 2$ , viscosity-driven “bar” mode instability, seem to suggest the opposite effect. BFG’s method consists of perturbing a stationary, axisymmetric configuration, obtained from a two dimensional numerical simulation, and retaining only the dominant terms in the nonaxisymmetric relativistic perturbation equations. BFG found that the critical adiabatic index for the bar instability becomes higher than the value of James as the configuration becomes increasingly relativistic. This behaviour suggests that relativistic effects tend to stabilize the configurations. Accordingly, SF concluded that, in general relativity, the point of onset of the viscosity-driven and the gravitational radiation-driven  $m = 2$  modes may no longer coincide as they do in Newtonian theory, and that the effect of relativity seems to be very different in the two cases. The main improvement presented in Bonazzola, Friebe & Gourgoulhon (1997) over their previous study consists in the fully three dimensional treatment of the shift vector. With this more detailed analysis, the stabilization of relativistic configurations is not only confirmed, but also strongly enhanced. Both SF’s and BFG’s findings are based on a numerical solution of perturbation equations and represent the only attempts to solve the relativistic problem in 3D to date.

In this paper we reconsider the problem of relativistic rotating equilibria and bar mode instabilities from an analytic point of view. Specifically, we extend the earlier Newtonian treatment of Lai, Rasio & Shapiro (1993a, hereafter LRS), which is based on triaxial ellipsoid models of rotating stars and an energy variational principle, to post-Newtonian (PN) gravitation. We restrict our discussion to incompressible, rigidly rotating bodies and neglect any deviation from the ellipsoidal shape in the equilibrium configuration. The ellipsoidal approximation is exact for rotating incompressible stars in Newtonian theory, but it is only approximate for PN configurations. However, our formalism allows us to derive the analytic functionals for the main global parameters characterizing a rotating configuration (total mass-energy, rest mass and

angular momentum). By applying a energy variational principle to these functionals, it is possible to construct equilibrium sequences of constant rest mass and to locate instability points along the sequence.

Our PN analysis is carried out in the framework of a 3+1 ADM splitting of the metric (Arnowitt, Deser and Misner 1962). In this respect, our solution sets up and should provide an important test-bed calculation for future numerical studies of 3D relativistic rotating configurations. Numerical relativity in 3D is only in its infancy, and lacks a large body of known solutions which it can attempt to reproduce. Part of the complication is related to the fact that, in general relativity, 3D stellar systems are usually fully dynamical due to the generation of gravitational waves. As a result, it is typically necessary to solve the full set of Einstein’s equations to determine the behavior of a 3D system. There are many asymmetric systems, however, in which the intrinsic dynamical timescale is much shorter than the timescale for dissipation due to gravitational waves. Such is the case for any weak field, slow velocity system, or for a strong-field, high velocity system that is only slightly perturbed from stationary equilibrium. It is thus possible to discuss “quasi-equilibrium” configurations for such objects. Examples include rotating neutron stars (NSs) either with small compaction  $M/R$  or small departure from axisymmetry, and binary neutron stars prior to reaching the innermost stable circular orbit (Baumgarte *et al.* 1997b). Following the discussion by Wilson & Matthews (1989, 1995) and Wilson (1990), Cook, Shapiro and Teukolsky 1992 (hereafter CST92) have provided a simplified set of 3+1, ADM equations which are well-adapted for studying quasi-equilibrium relativistic systems. For strong-field objects, these equations yield precise solutions to the initial value (constraint) equations, and approximate instantaneous snapshots of the object as it evolves, due to the emission of gravitational waves, provided it does so slowly. In this paper, we adopt the CST92 equations which are well suited to future numerical studies of the fully nonlinear equations for strong-field, quasi-equilibrium sources. We solve these equations in the PN approximation; at this order, the CST92 equations are exact. In this paper, we focus on the viscosity-driven, secular instability with respect to the formation of a barlike structure in a PN incompressible, rotating star. We find that PN solutions are unstable at rotation rates greater than in the Newtonian limit. Thus, in the PN treatment, nonaxisymmetric instabilities driven by viscosity set a less stringent limit on the maximum angular velocity than suggested by Newtonian theory.

From the observational point of view, the issue of investigating the maximum spin velocity for a NS is manifold. Rotating configurations subject to nonaxisymmetric instabilities could become important sources of gravitational waves emission and represent possible candidates for detection by laser interferometers now under construction, like LIGO, GEO and VIRGO (see e.g. Bonazzola & Marck 1994; Thorne 1987; Lai & Shapiro 1995; Schutz 1997 and reference therein). There are a number of plausible astrophysical situations in which this effect can set in. Numerical simulations of coalescing binary neutron stars show that merger into a single object is the probable end point of close binary evolution (see e.g. Rasio & Shapiro 1992, 1994; LRS; Lai, Rasio & Shapiro 1993b, 1994a,b,c). The newly born object may be rotating quite rapidly (Baumgarte *et al.* 1997b) and might be driven secularly unstable. Core collapse in massive, evolved stars or accretion-induced collapse of white dwarfs, also has been suggested as observable sources of gravitational radiation (Lai & Shapiro 1995 and references therein). Coalescing white dwarf binaries are thought to be progenitors of type Ia supernovae (Iben & Tutukov 1984; Yungelson *et al.* 1994), or, in certain cases, of isolated millisecond pulsars (Chen & Leonard 1993). In all these scenarios, the newly born NS can undergo a secular or dynamical evolution, breaking its axisymmetry, provided that its spin velocity is larger than the critical value. Alternatively, accretion onto NSs in X-ray binary systems or in Thorne–Zytkow objects can spin up the compact star until reaching the critical value of  $T/|W|$ , at which point the Chandrasekhar-Friedman-Schutz (CFS) instability (Chandrasekhar 1970; Friedman & Schutz 1978) may drive a nonaxisymmetric mode unstable, powering radiation. This “forced gravitational emission”, is

particularly appealing for maintaining steady, periodic emission, with the total amount of accreted angular momentum balancing the amount that is radiated away via gravitational radiation (Wagoner 1984; Schutz 1997).

We point out that our analytical results assume incompressible and rigidly rotating bodies, and both assumptions have been introduced mainly to make an analytical treatment tractable, to PN order. However, for applications to NSs, we note that both assumptions are not entirely “ad hoc”. Viscosity tends to drive NS to rigid rotation (Friedmann & Ipser 1987). In addition, recent many body calculations suggest that the EOS of dense nuclear matter is relatively stiff (see, e.g. Wiringa, Fiks & Fabronici 1988 and references therein). In view of these considerations, application of our results to realistic situations may not be unreasonable, at least as a first approximation.

In section 2, we review the physical problem and summarize the results of previous investigations. The variational principle of LRS is reviewed in section 3, while in section 4 we set up the mathematical problem in the PN approximation. We assemble the relevant set of 3+1 equations for the gravitational field and present analytic solutions for both the metric coefficients and the global conserved quantities. These results are then used to build equilibrium sequences in section 5 and locate the bar mode instability point in section 6. Discussion and conclusions follow in section 7.

## 2. Previous Investigations

The problem of the equilibrium shape and stability of an incompressible, rigidly rotating configuration admits an exact analytic solution in Newtonian gravitation (Ch69). In this case the rotating configuration takes in axisymmetry the equilibrium shape of a Maclaurin spheroid. However, nonaxisymmetric instabilities can develop in rapidly spinning spheroids when the ratio  $T/|W|$  of the rotational kinetic to the gravitational potential energy becomes sufficiently large. At the critical value  $T/|W| = 0.1375$  the equilibrium sequence of MacLaurin configurations bifurcates into two other branches of triaxial equilibria, the Jacobi and the Dedekind ellipsoids. Since the Maclaurin spheroids are dynamically unstable only for  $T/|W| > 0.2738$ , the bifurcation point is dynamically stable. However, in presence of a suitable dissipative mechanism such as viscosity or gravitational radiation, this point becomes secularly unstable to a  $l = 2$   $m = 2$  bar mode. Secular instability may play an important role in limiting the maximum rotation of neutron stars. Viscosity dissipates the mechanical energy  $E$ , but preserves angular momentum  $J$ . Consequently, a Maclaurin spheroid undergoing a viscosity driven instability terminates its evolution as a Jacobi ellipsoid, with a lower value of  $E$  but the same value of  $J$  (and rest mass  $M_0$ ). The Jacobi solution is rigidly rotating, so that the viscous dissipation stops once it is formed. Alternatively, the Dedekind ellipsoid represents a lower energy state (with respect to the Maclaurin solution) for a given circulation. A rapidly spinning configuration can be unstable to the emission of gravitational radiation at the bifurcation point (the CFS instability). Since this process does not conserve angular momentum but conserves circulation, under a CFS instability the growth of the bar mode leads to the deformation of a Maclaurin spheroid into a Dedekind ellipsoid. At this final state, which is stationary, the emission of gravitational waves stops. The competition between the Jacobi-like and the Dedekind-like modes is governed by the ratio of the strength of the viscous stress to the gravitational radiation reaction force, and this ratio depends crucially on the internal properties of the star. Although the determination of a realistic equation of state for NSs and, as a consequence, of the strength of the viscosity, is still an open issue, viscosity is thought to be more important for old NSs and gravitational radiation in the hottest, newly born objects (see e.g. BFG and references therein). The situation is more complicated when both viscosity and gravitational radiation act together, since they tend to cancel each

other, stabilizing the star (Lindblom & Detweiler 1977; Lai & Shapiro 1995).

A number of efforts in Newtonian physics have been devoted to the extension of the Ch69’s results to more realistic, compressible fluids, modeled by a polytropic equation of state. For rigidly rotating polytropes, bifurcation to triaxial configurations can only exist when the adiabatic index exceeds a critical value,  $\gamma_{crit} = 2.238$  (Jeans 1919, 1928; James 1964). This is because the EOS must be stiff enough to make the angular velocity at the bifurcation point lower than the limiting value  $\Omega_k$  at which the centrifugal force balance the gravitational force at the equator (mass shedding limit). Moreover, Ipser & Managan (1985) demonstrated that the  $m = 2$  Jacobi-like bifurcation point and the  $m = 2$  Dedekind-like point have the same location along uniformly rotating, polytropic sequences, as in the incompressible case (see also LRS; Lai & Shapiro 1995). Typically, rotating equilibrium stellar models must be constructed numerically. Models have been constructed by a number of authors, using both polytropic (see e.g. Bodenheimer & Ostriker 1973; Ipser & Managan 1981; Hachisu & Eriguchi 1982; Hachisu 1986a,b; see Tassoul 1978 for an extensive set of references) and more realistic equations of state for both white dwarfs and NSs (see e.g. Ostriker & Tassoul 1969; Durisen, 1975; Hachisu 1986a; BFG and references therein).

LRS constructed triaxial ellipsoid models of rotating polytropic stars in Newtonian gravity, using an ellipsoidal energy variational method. This approach was originally introduced by Zel’dovich & Novikov (1971, see also Shapiro & Teukolsky 1983, hereafter ST) in the axisymmetric case, to investigate the stability of a polytropic star against gravitational collapse. The main advantage of the method comes from its simplicity: all results are analytic or quasi-analytic, lending themselves to straightforward physical interpretation. This is in part because the method deals directly with global, conserved quantities. When quantities like the total energy and the total angular momentum are determined along an equilibrium sequence, the evolution of the system can be tracked as it loses  $E$  or  $J$  by some quasi-static dissipative process. Generalizing the same approach to the triaxial case, LRS were able to construct equilibrium sequences for compressible analogues of most classical incompressible objects, like isolated Maclaurin, Jacobi, Dedekind and Riemann ellipsoids and binary Roche, Darwin and Roche-Riemann ellipsoids.

Neutron stars are relativistic objects and the analysis of their equilibrium and stability must be based necessarily on general relativistic models. The structure of a rotating axisymmetric star in general relativity has been investigated numerically by a number of authors (see e.g. Butterworth & Ipser 1976, hereafter BI; Friedman, Ipser & Parker 1986; CST92; Cook, Shapiro & Teukolsky 1994a,b; Cook, Shapiro & Teukolsky 1996, hereafter CST96 and references therein), but the first fully relativistic computations of nonaxisymmetric instabilities have been presented only very recently (SF; Stergioulas 1997). From the analytical point of view, even less is known. An exact treatment of the radial oscillations of a gaseous mass in general relativity is possible (Chandrasekhar 1964), but a similar analysis of nonradial oscillations is not to be expected. In fact, apart from the difficulties associated with the solution of the Einstein equations without any presupposed symmetry, allowance must be made in the relativistic regime for the emission of gravitational waves. However, some insight into the nature of the general relativistic effects can be obtained by examining the problem in the PN approximation, i.e. at a level in which gravitational radiation plays no role. PN effects on the equilibrium of uniformly rotating, homogeneous bodies have been extensively investigated in a series of paper by Chandrasekhar (1965a,b, 1967a,b,c, 1969b; see also Chandrasekhar & Nutku 1969, hereafter CN, for the PPN corrections) using the tensor virial formalism. Whenever possible in this paper, we make a direct comparison between our expressions and the corresponding PN results derived by Chandrasekhar. In particular, starting from the PN equations of hydrodynamics, he derived the equilibrium relation between the eccentricity and the angular velocity. Although he obtained integral expressions for the global conserved quantities, he did not evaluate them or give explicit formulae for the

PN corrections to the rest-mass, angular momentum and binding energy. A different method, based on the solution of the PN Poisson equation in oblate spheroidal coordinates, was presented by Bardeen (1971). The formalism turns out to be simpler with respect to the Chandrasekhar one, but this approach does not allow an immediate generalization to the triaxial case. In 3D, the general eigenfunctions resulting from the separation (Lame functions) are indeed available, but many details about their general properties are not well studied. No analytic investigations of the location of the secular stability point in general relativity were provided in these earlier studies or, to our knowledge, elsewhere in the literature.

### 3. The Energy Variational Method

In this section we briefly review the energy variational approach in Newtonian theory (LRS), to introduce the basic concepts that we use in our PN calculations.

Consider a self-gravitating, isolated system. Each configuration (whether or not in equilibrium) can be specified by the total energy  $E$  and a number of conserved global quantities, such as the rest mass  $M_0$  and the total angular momentum  $J$ . Since  $E$  can be always written in terms of the fluid density and velocity fields  $\rho(\mathbf{x})$ ,  $v(\mathbf{x})$

$$E = E[\rho(\mathbf{x}), v(\mathbf{x}); M_0, J, \dots], \quad (1)$$

the equilibrium state can be determined by extremizing this functional with respect to all variations in both  $\rho(\mathbf{x})$  and  $v(\mathbf{x})$ , under the constraint that the conserved quantities are unchanged. Direct application of such a variational approach to a multidimensional system is a computationally challenge task. However, as discussed by LRS, a great simplification arises when we can replace the infinite number of degrees of freedom contained in  $\rho(\mathbf{x})$  and  $v(\mathbf{x})$  by a limited number of free parameters  $\alpha_1, \alpha_2, \dots$ . This we can often do for sufficiently simple systems under suitable simplifying assumptions. The total mass energy then becomes

$$E = E[\alpha_1, \alpha_2, \dots; M_0, J, \dots] \quad (2)$$

and the equilibrium configuration is determined by extremizing this functional according to

$$\frac{\partial E}{\partial \alpha_i} = 0 \quad i = 1, 2, \dots \quad (\text{equilibrium}) \quad (3)$$

under the constraint that  $M_0, J, \dots$  are conserved.

The onset of the instability can be then determined from

$$\det \left( \frac{\partial^2 E}{\partial \alpha_i \partial \alpha_j} \right)_{eq} = 0, \quad i, j = 1, 2, \dots, \quad (\text{onset of instability}) \quad (4)$$

where the subscript ‘eq’ indicates quantities evaluated along the equilibrium sequence. Clearly, whether the instability actually arises depends on the presence of a suitable dissipative mechanism which preserves the conservation laws assumed in the construction of the equilibrium model.

Now consider a simple application: a homogeneous, uniformly rotating, Newtonian fluid system, with density  $\rho_0$  and angular velocity  $\Omega$ . In the incompressible case, the internal energy vanishes and the total energy is given by

$$E = T + W, \quad (5)$$

where  $T$  and  $W$  are the rotational kinetic and the gravitational contributions, respectively. Assume that the surface of the configuration is ellipsoidal in shape. Then the geometry of the system is completely

specified by the values  $a_i$  ( $i = 1 \dots 3$ ) of the three semiaxes of the outer surface, where the pressure vanishes. However, following LRS, it is more convenient to introduce an equivalent set of parameters defined by

$$\lambda_1 = \left( \frac{a_3}{a_1} \right)^{2/3}, \lambda_2 = \left( \frac{a_3}{a_2} \right)^{2/3}, \quad (6)$$

and

$$R = (a_1 a_2 a_3)^{1/3}. \quad (7)$$

Note that  $R$  represents the radius of the spherical configuration with the same volume as the rotating one; it is not, in general, related to the equilibrium state. The gravitational potential and the kinetic energies can be written as

$$W = -\frac{3}{5} \frac{GM_0^2}{R} \frac{I_{Ch}}{2(a_1 a_2 a_3)^{2/3}} = -\frac{3}{5} \frac{GM_0^2}{R} f, \quad (8)$$

$$T = \frac{J^2}{2I} = \frac{J^2}{2I_s} h, \quad (9)$$

where  $M_0 = 4\pi\rho_0 R^3/3$ ,  $J = \Omega I$  and  $I_s = 2M_0 R^2/5$  is the momentum of inertia of a sphere of the same volume. The momentum of inertia of the ellipsoid is  $I = I_s/h$ , and the two dimensionless ratios  $f$  and  $h$  are defined as

$$f = \frac{I_{Ch}}{2R^2} = \frac{1}{2} \left( \frac{A_1 \lambda_2}{\lambda_1^2} + \frac{A_2 \lambda_1}{\lambda_2^2} + A_3 \lambda_1 \lambda_2 \right), \quad (10)$$

$$h = \frac{2R^2}{a_1^2 + a_2^2} = \frac{2\lambda_1^2 \lambda_2^2}{\lambda_1^3 + \lambda_2^3}. \quad (11)$$

Here  $I_{Ch} = \sum_i A_i a_i^2$  [called  $I$  in Ch69, equation (3.15)], and the dimensionless coefficients  $A_i$  can be calculated in terms of standard incomplete elliptic integrals involving only the axis ratios [see equations (3.33)–(3.35) in Ch69]. In the spherical limit  $a_1 = a_2 = a_3 = R$  and  $f = h = 1$ .

The total mass–energy may now be written as

$$E = -\frac{3}{10} \frac{M_0^2}{R^3} (I_{Ch} + th), \quad (12)$$

where

$$t = -\frac{5}{3} \frac{J^2 R^3}{I_s M_0^2} = -\frac{1}{4} \frac{\Omega^2}{\pi \rho_0} \frac{a_1^2 + a_2^2}{h}. \quad (13)$$

The equilibrium sequence can be constructed by varying  $E$  according to

$$\frac{\partial E}{\partial \lambda_1} = \frac{\partial E}{\partial \lambda_2} = 0, \quad (14)$$

while holding  $M_0$ ,  $J$  constant. For an incompressible fluid  $\rho_0 = \text{constant}$  and there no variations with respect to  $R$ . This gives

$$\begin{aligned} 0 &= \frac{\partial E}{\partial \lambda_1} = -\frac{3}{10} \frac{M_0^2}{R^3} \left( \frac{\partial I_{Ch}}{\partial \lambda_1} + t \frac{\partial h}{\partial \lambda_1} \right) \\ 0 &= (1 \leftrightarrow 2). \end{aligned} \quad (15)$$

The derivatives of  $h$  and  $I_{Ch}$  with respect to  $\lambda_1$  and  $\lambda_2$  are reported in LRS [equations (A3), (A9)]. Exploiting these formulas and using expression (13), conditions (15) can be cast in the form

$$\begin{aligned} 0 &= \frac{3}{2} a_1^2 A_1 - \frac{1}{2} I_{Ch} - \frac{1}{4} \frac{\Omega^2}{\pi \rho_0} (2a_1^2 - a_2^2) \\ 0 &= (1 \leftrightarrow 2), \end{aligned} \quad (16)$$

and these can be combined by adding each one to 1/2 times the other, yielding

$$\frac{\Omega^2}{\pi\rho_0}a_1^2 - 2a_1^2A_1 = \frac{\Omega^2}{\pi\rho_0}a_2^2 - 2a_2^2A_2 = -2a_3^2A_3. \quad (17)$$

From Ch69, we write

$$A_{ij} = A_{ji} = \frac{A_i - A_j}{a_j^2 - a_i^2} \quad (i \neq j) \quad (18)$$

$$B_{ij} = B_{ji} = A_j - a_i^2 A_{ij} \quad (19)$$

$$2 = 3A_{ii}a_i^2 + A_{ij}a_i^2 + A_{ik}a_i^2 \quad (i \neq j \neq k) \quad (20)$$

and add to each side of (17) the quantity  $2a_1^2a_2^2A_{12}$ , obtaining

$$a_1^2 \left( \frac{\Omega^2}{\pi\rho_0} - 2B_{12} \right) = a_2^2 \left( \frac{\Omega^2}{\pi\rho_0} - 2B_{12} \right) = 2(a_1^2a_2^2A_{12} - a_3^2A_3). \quad (21)$$

The latter equalities allow a solution with  $a_1 \neq a_2$  if and only if

$$a_1^2a_2^2A_{12} = a_3^2A_3, \quad (22)$$

$$\frac{\Omega^2}{\pi\rho_0} = 2B_{12}, \quad (\text{Jacobi ellipsoid}) \quad (23)$$

and, as it is well known, these two conditions determine the equilibrium sequence of Jacobi ellipsoids. In particular, the first represents a relation between the two axial ratios, while the latter gives the angular velocity. When  $a_1 = a_2$ , we obtain

$$a_1^4A_{11} = a_3^2A_3, \quad \frac{\Omega^2}{\pi\rho_0} = 2B_{11}, \quad (24)$$

or

$$\frac{\Omega^2}{\pi\rho_0} = 2 \left( A_1 - \frac{a_3^2}{a_1^2} A_3 \right), \quad (\text{Maclaurin spheroid}) \quad (25)$$

which recovers the Maclaurin sequence. To determine the condition for the onset of the secular instability to nonaxisymmetric perturbations (LRS), we solve

$$\det \left( \frac{\partial^2 E}{\partial \lambda_i \partial \lambda_j} \right)_{eq} = 0, \quad i, j = 1, 2. \quad (26)$$

Since we have

$$\left( \frac{\partial^2 E}{\partial \lambda_1^2} \right)_{eq} = \left( \frac{\partial^2 E}{\partial \lambda_2^2} \right)_{eq} \quad (27)$$

the determinant vanishes at the two points

$$\left( \frac{\partial^2 E}{\partial \lambda_1^2} \right)_{eq} = \pm \left( \frac{\partial^2 E}{\partial \lambda_1 \partial \lambda_2} \right)_{eq}. \quad (28)$$

However, only the plus sign is relevant: it is easy to demonstrate that the minus sign corresponds to a negative value of the ratio  $\Omega^2/(\pi\rho_0)$  and must be discarded. The relevant solution gives the condition

$$\left( \frac{\partial^2 I_{Ch}}{\partial \lambda_1^2} \right)_{eq} + t \left( \frac{\partial^2 h}{\partial \lambda_1^2} \right)_{eq} = \left( \frac{\partial^2 I_{Ch}}{\partial \lambda_1 \partial \lambda_2} \right)_{eq} + t \left( \frac{\partial^2 h}{\partial \lambda_1 \partial \lambda_2} \right)_{eq}. \quad (29)$$



Finally, using the equations of the second derivatives reported in LRS [equations A(3), (A9)], equation (29) can be cast in the form

$$\frac{\Omega^2}{\pi\rho_0} = 2B_{11}, \quad (30)$$

which is the exact expression for the secular instability point in Maclaurin spheroids with respect to nonaxisymmetric perturbations (see Ch69, LRS). Given the constraint that the deformations be ellipsoidal, we have identified the onset of the bar mode ( $m = 2$ ) instability point. Here the instability we have located would be triggered by the presence of viscosity, since uniform rotation was assumed in building the equilibrium sequence and maintained in the variations, while holding fixed  $M_0$  and  $J$  (but not circulation). We can also recover the result that this condition occurs at the precise point where the Jacobi sequence bifurcates from the Maclaurin sequence [see (23), (24)].

Using the variational method in the framework in Newtonian physics, LRS were able to investigate a number of problems. In particular, they constructed approximate hydrostatic equilibrium solutions for rotating polytropes, either isolated or in binary systems, and presented the compressible generalization of the classical sequences of Maclaurin spheroids, Jacobi, Dedekind and Riemann ellipsoids and Roche, Darwin and Roche–Riemann binaries. For the case of incompressible Maclaurin sequences the ellipsoidal approximation is exact and the results derived from the variational principle are exact (Ch69). In this paper we will use the same variational approach to investigate the ellipsoidal configurations in PN gravitation.

#### 4. The Post–Newtonian Solution

We will construct a variational expression for the total mass–energy  $M$  of a rotating ellipsoid in PN gravitation. The variation in  $M$  is equivalent to a variation in  $E = M - M_0$  for fixed  $M_0$ , and we will consider incompressible configurations with constant rest mass density  $\rho_0$ . For a PN treatment the evaluation of all the integral quantities (e.g.  $M$ ,  $M_0$  and  $J$ ) requires a knowledge of the metric, which must be determined self-consistently with the matter profile by solving the Einstein field equations. The field equations form a set of coupled, nonlinear partial differential equations and, in general, must be solved numerically. The solution of these equations for a rotating star in general relativity has been tackled by many authors (see e.g. CST92; Cook, Shapiro & Teukolsky 1994a,b; CST96, and references therein). A great simplification arises if we adopt the “conformal approximation” of CST96 (see also Wilson & Matthews 1989, 1995; Wilson 1990, Wilson, Matthews & Marronetti 1996). This form for the metric is only approximate (although the approximation is very good) for rotating stars in full general relativity (CST96), for two main reasons. First, even in absence of gravitational radiation (e.g., the case for stationary, axisymmetric equilibrium rotating stars) the exact solution is not conformal. However, as shown by CST96, the deviation from conformal flatness is small ( $< 1\%$  even for highly relativistic stars). The conformal approximation is exact in PN gravitation and departures only enter at the 2 PN order. The second reason is that emission of gravitational radiation causes the system to evolve in time, so the conformal decomposition of CST96 does not hold in general. However, the timescale for the evolution due to gravitational radiation is much longer than the dynamical timescale, and in many situations relativistic systems can be treated to be in quasi-equilibrium (see e.g. Baumgarte *et al.* 1997a,b). An analogous approximation is often used in stellar evolution calculations, where the relevant evolution timescales are the nuclear or Kelvin–Helmholtz timescales, so that the stars maintain (quasi) hydrostatic equilibrium on a dynamical timescale. Gravitational radiation only enters at the 2.5 PN order. Henceforth, we restrict our attention to configurations in PN theory, where the 3–metric is conformally flat. We choose to use the 3+1 decomposition of Einstein’s equations in our analysis, rather than the conventional Chandrasekhar

(1965a) PN expansion. We do so in preparation for future numerical treatments of nonaxisymmetric instabilities and their nonlinear evolution, which are best tracked in 3+1 form (see also Bonazzolla, Friebe and Gourgoulhon 1997).

Henceforth we confine our attention to the case of incompressible, rigidly rotating models, for which the PN system can be solved analytically provided we adopt a ellipsoidal model for the matter profile.

We solve the conformal ADM equations for the metric in this section. We introduce our approximations, section 4.1, referring to CST92 and CST96 for the notation and for a more detailed discussion. The final metric is presented in section (4.2) and expressions for the conserved quantities are derived in (4.3). Our field equations are equivalent to those of CN in the PN limit.

#### 4.1. Basic Equations

Let us consider an isolated, self-gravitating, homogeneous system with rest mass density  $\rho_0$  and total mass-energy density  $\rho$ . Assume that it is uniformly rotating with a constant angular velocity  $\Omega$ . In the incompressible limit, the internal energy is zero, and the energy-momentum tensor takes the form

$$T_{\mu\nu} = (\rho_0 + P) U_\mu U_\nu + P g_{\mu\nu}, \quad (31)$$

where  $U^\mu$  is the fluid four-velocity,  $g_{\mu\nu}$  are the metric coefficients and  $P$  is the pressure. Here Greek indices  $\mu, \nu, \dots$  range over  $0 \dots 3$ , while Latin indices  $i, j, \dots$  range over  $1 \dots 3$ ; geometrized units ( $c = G = 1$ ) are used throughout. We assume that the outer surface (where  $\rho = 0$ ) is a triaxial ellipsoid with semiaxes  $a_1, a_2, a_3$ .<sup>3</sup> Following CST96, we start from the most general expression for the metric in a 3+1 form

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij} (dx^i + \beta^i dt) (dx^j + \beta^j dt), \quad (32)$$

where  $\alpha$  and  $\beta^i$  are the lapse and the shift functions, respectively. We choose a conformally flat decomposition of the spatial metric

$$\gamma_{ij} = \Psi^4 f_{ij}. \quad (33)$$

Here  $\Psi$  is the conformal factor and  $f_{ij}$  is the Euclidean metric in the adopted coordinate system. In the following we will use cartesian coordinates  $x_i$ ,  $i = 1 \dots 3$ . Following CST96, we set  $\partial_t (\gamma^{-1/3} \gamma_{ij}) = 0$  and adopt the maximal slicing condition  $K = K_i^i = 0$ , where  $K_{ij}$  is the extrinsic curvature. We obtain [see equation (4) by CST96]

$$K_{ij} = \frac{1}{2\alpha} \left( D_i \beta_j + D_j \beta_i - \frac{2}{3} \gamma_{ij} D_k \beta^k \right). \quad (34)$$

Here  $D_i$  indicates the covariant derivative with respect to  $\gamma_{ij}$ . The metric coefficients depend on the three functions  $\alpha$ ,  $\beta^i$  and  $\Psi$  that, following CST96, can be derived as a solution of a system of ADM partial differential equations [see equations (8), (14) and (18) in CST96]. The first two equations are the Hamiltonian constraint equation and the lapse equation ( $\partial_t K = 0$ )

$$\nabla^2 \Psi = -\frac{1}{8} \Psi^5 K^{ij} K_{ij} - 2\pi \Psi^5 \rho, \quad (35)$$

---

<sup>3</sup>Here the ellipsoid is defined in coordinate space, so that the proper shape of the PN configuration is not in general ellipsoidal. Although convenient mathematically, such a trial function is only an approximation (except in the Newtonian limit, where it is exact).

$$\nabla^2 (\alpha \Psi) = (\alpha \Psi) \left[ \frac{7}{8} \Psi^4 K_{ij} K^{ij} + 2\pi \Psi^4 (\rho + 2S) \right], \quad (36)$$

while the differential equation for the shift vector can be obtained by substituting expression (34) into the momentum constraint

$$D_i K^{ij} = 8\pi S^j. \quad (37)$$

The final result is

$$\begin{aligned} \nabla^2 \beta^i + \frac{1}{3} \nabla^i (\nabla_j \beta^j) &= \left( \frac{1}{\alpha} \nabla_j \alpha - \frac{6}{\Psi} \nabla_j \Psi \right) \left( \nabla^j \beta^i + \nabla^i \beta^j - \frac{2}{3} f^{ij} \nabla_k \beta^k \right) \\ &+ 16\pi \alpha \Psi^4 S^i. \end{aligned} \quad (38)$$

In the previous expressions  $\nabla_i$  and  $\nabla^2$  denote the flat space covariant derivative and the Laplacian operator. The density  $\rho$  appearing into equations (35), (36) can be derived from the stress–energy tensor

$$\rho = n^\mu n^\nu T_{\nu\mu} = (\rho_0 + P) (\alpha U^t)^2 - P, \quad (39)$$

where  $n^\mu$  is the normal vector to a  $t = \text{constant}$  surface. Note that  $\rho = \rho_0$  for a nonrotating sphere, since there is no internal energy. However, the two densities differ in general and  $\rho$  is not a constant, due to rotational energy contributions. The source term  $S$  and the momentum source  $S^i$  are given by

$$S = \gamma^{ij} T_{ij} = (\rho_0 + P) \left[ (\alpha U^t)^2 - 1 \right] + 3P, \quad (40)$$

$$S^i = -\gamma_j^i n_k T^{jk} = (\rho_0 + P) (\alpha U^t) \gamma^{ij} U_j. \quad (41)$$

Equation (38) can be conveniently reduced to two simpler equations by introducing the decomposition

$$\beta^i = G^i - \frac{1}{4} \nabla^i B. \quad (42)$$

The two equations that must be solved now become

$$\nabla^2 G^i = \left( \frac{1}{\alpha} \nabla_j \alpha - \frac{6}{\Psi} \nabla_j \Psi \right) \left( \nabla^j \beta^i + \nabla^i \beta^j - \frac{2}{3} f^{ij} \nabla_k \beta^k \right) + 16\pi \alpha \Psi^4 S^i, \quad (43)$$

$$\nabla^2 B = \nabla_k G^k. \quad (44)$$

In the fully relativistic case the solution of the coupled ADM equations (35), (36), (43) and (44) represents a nontrivial problem and must be tackled numerically. In this paper we work at the PN order and, in the ellipsoidal approximation, the equations can be solved analytically. The metric and the stress tensor will be expanded as sums of terms of successively higher order in the expansion parameter  $1/c^2$ , while each ADM equation will be decomposed into a series of equations of successively highly order in  $1/c^2$ . The first PN correction will refer to the terms that are  $O(c^{-2})$  (i.e.,  $O(M/R)$ , where  $R$  is a length scale of the problem) higher than the corresponding Newtonian terms in this expansion. We do not restrict our analysis to slow rotation (Hartle 1967), whereby one requires  $\sqrt{\frac{R^3}{M}} \Omega \ll 1$ . In that context, rotation is considered “slow” if its effects on the structure of the star are relatively small. Here we allow arbitrary fast rotation, so that  $\Omega^2$  is permitted to reach  $\sim (M/R^3)$  and stars can suffer considerable rotational distortion.

To obtain equations correct to the PN order, in the right–hand sides of (35), (36) and (38) we need retain only the contributions of order

$$\rho_0, \quad \rho_0 \frac{M}{R}. \quad (45)$$

Consider the Hamiltonian constraint and the lapse equations. From equations (38) and (41) we have, at the leading order,

$$S^i \sim \rho_0 v \quad \Rightarrow \quad \beta^i \sim S^i R^2 \sim \rho_0 v R^2 \sim v \frac{M}{R}, \quad (46)$$

which yields [see equation (34)]:

$$K_{ij} K^{ij} \sim \left( \frac{\beta^i}{R} \right)^2 \sim \rho_0 \frac{M^2}{R^2}. \quad (47)$$

This means that, at our order of approximation, we can safely drop all terms involving the extrinsic curvature from equations (35) and (36), which then reduce to the simpler form

$$\nabla^2 \Psi = -2\pi \Psi^5 \rho, \quad (48)$$

$$\nabla^2 (\alpha \Psi) = 2\pi \alpha \Psi^5 (\rho + 2S). \quad (49)$$

We thus need approximate expressions for the source terms  $\rho$  and  $S$ , and these can be obtained by expanding the product  $(\alpha U^t)$  appearing in (39) and (40). The normalization condition  $U_\nu U^\nu = -1$  yields

$$(\alpha U^t)^2 = 1 + \gamma^{ij} U_i U_j. \quad (50)$$

We introduce  $v^i \equiv U^i/U^t$  and consider a velocity field corresponding to uniform rotation with angular velocity  $\Omega = U^\phi/U^t$  about the  $x_3$  direction. Then

$$v_1 = -\Omega x_2, \quad v_2 = \Omega x_1, \quad v_3 = 0, \quad (51)$$

and  $v^2 = \Omega^2(x_1^2 + x_2^2)$ . At the leading term, it follows that

$$(\alpha U^t)^2 \approx 1 + v^2 + \dots \quad (52)$$

Substituting back in the definition of  $\rho$  and  $S$  and recalling that  $P/\rho_0 \sim O(M/R)$ , we find that, at Newtonian order, these two quantities can be approximated as

$$\rho \approx \rho_0 (1 + v^2 + \dots), \quad (53)$$

$$S \approx \rho_0 \left( v^2 + 3 \frac{P}{\rho_0} + \dots \right). \quad (54)$$

We also need the pressure distribution in the equilibrium configuration. Terms involving  $P$  appear only in the PN correction. This means that we can use the Newtonian result for  $P$  (see e.g. Chandrasekhar 1965a, hereafter Ch65a)

$$\frac{P}{\rho_0} = \pi \rho_0 \left[ A_3 a_3^2 - \sum_i A_i x_i^2 \right] + \frac{1}{2} \Omega^2 (x_1^2 + x_2^2). \quad (55)$$

The functions  $\Psi$  and  $\alpha$  now can be derived by solving the two equations (48) and (49). The conformal factor can be expanded as:

$$\Psi \equiv 1 - \Phi/2, \quad (56)$$

where  $\Phi \equiv \Phi_N + \Phi_{PN}$  and where

$$\nabla^2 \Phi_N \equiv 4\pi \rho_0. \quad (57)$$

Note that  $\Phi_N$  represents the Newtonian potential and coincides with the quantity  $-U$  in the Chandrasekhar's notation. Linearizing  $\Psi^5 \approx (1 - 5\Phi/2)$  and substituting (53), (56), (57) into (48), we obtain

$$\nabla^2 \Phi_{PN} = -10\pi \rho_0 \Phi_N + 4\pi \rho_0 v^2. \quad (58)$$

In a similar way, the lapse equation can be linearized by introducing the expansion  $\alpha\Psi \equiv 1 + \Theta$ , where  $\Theta \equiv \Theta_N + \Theta_{PN}$  and  $\nabla^2\Theta_N \equiv 2\pi\rho_0$ , so that  $\Theta_N = \Phi_N/2$ . Writing

$$\Psi^4 \approx \left(1 - \frac{\Phi_N}{2} \dots\right)^4 \approx 1 - 2\Phi_N \dots, \quad (59)$$

and

$$\rho + 2S \approx \rho_0 \left(1 + 3v^2 + 6\frac{P}{\rho_0} \dots\right), \quad (60)$$

we obtain

$$\nabla^2\Theta_{PN} = 6\pi\rho_0 \left(v^2 + 2\frac{P}{\rho_0} - \frac{1}{2}\Phi_N\right). \quad (61)$$

The corresponding expansion of the lapse function is

$$\alpha = (1 + \Theta)\Psi^{-1} \approx 1 + \frac{\Phi_N}{2} + \frac{\Phi_{PN}}{2} + \frac{\Phi_N^2}{4} + \Theta_N + \Theta_N \frac{\Phi_N}{2} + \Theta_{PN}. \quad (62)$$

It is straightforward to identify the Newtonian and the PN contributions to  $\alpha$  as

$$\alpha_N = \Phi_N, \quad (63)$$

$$\alpha_{PN} = \frac{\Phi_N^2}{2} + \frac{\Phi_{PN}}{2} + \Theta_{PN}. \quad (64)$$

Parenthetically, we note that  $\alpha_N$  and  $\alpha_{PN}$  coincide with the correspondent terms derived by CN (see Appendix A.2). Summarizing, to derive the PN corrections to the lapse function and the conformal factor, we need to solve the system of elliptic equations (57), (58) and (61).

Let us now focus on the shift vector, and introduce the expansions

$$\beta^i = \beta_{PN}^i + O(c^{-5}), \quad G^i = G_{PN}^i + O(c^{-5}), \quad B = B_{PN} + O(c^{-5}). \quad (65)$$

At our order of approximation, we only need retain the leading terms in these expansions, of order  $\sim vM/R \sim c^{-3}$ . As a consequence, we can legitimately drop the first term on the right hand of equation (43), being of order

$$\frac{M}{R^2} \frac{\beta^i}{R}, \quad (66)$$

(note that  $\nabla\alpha \sim \nabla\Phi \sim M/R^2$ ). This yields

$$\nabla^2 G_{PN}^i \approx 16\pi\alpha\Phi^4 S^i \approx 16\pi S^i, \quad (67)$$

and we are left to derive the leading term in  $S^i$ . In expression (41) we have

$$\begin{aligned} \gamma^{ij}U_j &= \gamma^{ij}g_{j\nu}U^\nu \\ &= U^t(\beta^i + v^i) \approx v^i, \end{aligned} \quad (68)$$

which yields to leading order

$$S^i \approx \rho_0 v^i. \quad (69)$$

By substituting expressions (51) for the components of  $v^i$ , we can finally write the two equations

$$\nabla^2 G_{PN}^1 = 16\pi\rho_0 v^1 = -16\pi\rho_0 \Omega x_2, \quad (70)$$

$$\nabla^2 G_{PN}^2 = 16\pi\rho_0 v^2 = 16\pi\rho_0 \Omega x_1, \quad (71)$$

while  $G_{PN}^3 = 0$ . Once  $G_{PN}^i$  is known,  $B_{PN}$  can be obtained simply by solving

$$\nabla^2 B_{PN} = \frac{\partial G_{PN}^1}{\partial x_1} + \frac{\partial G_{PN}^2}{\partial x_2}, \quad (72)$$

and  $\beta_{PN}^i$  will follow from (42).

As it will be shown in the next section, the full system of elliptic equations (57), (58), (61), (70)–(72) admit an analytical solution (up to well known elliptic integrals).

#### 4.2. The Metric: Analytic Solution

The Newtonian potential  $\Phi_N$  at an internal point  $x_i$  of a homogeneous, ellipsoidal configuration with semi-axes  $a_i$  has the well known analytical form

$$\Phi_N = -\rho_0 \int \frac{1}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}' = -\pi\rho_0 \left( I_{Ch} - \sum_i A_i x_i^2 \right), \quad (73)$$

[see e.g. Ch69, equation (3.40)]. This function provides the leading term in the expansions for  $\Psi$  and  $\alpha$  [equations (56), (63)]. Hence we only need to determine the post-Newtonian corrections  $\Phi_{PN}$ ,  $\alpha_{PN}$  and  $\beta_{PN}^i$  to obtain the metric at the PN order. The main advantage of our decomposition comes from the fact that the PN contributions can all be written in integral form by exploiting the Green's functions of the corresponding differential equations. Moreover, in our homogeneous, rigidly rotating case, the aforementioned integrals involve familiar quadratures of the kind

$$\rho_0 \int \frac{1}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}', \quad \rho_0 \int \frac{x_i}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}', \quad \rho_0 \int \frac{x_i x_j}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}'. \quad (74)$$

The first one is simply  $-\Phi_N$ , while the others coincide with the two Newtonian potentials  $D_i$  and  $D_{ij}$  given by Ch69 [see equations (3.120), (3.131)]. As a result, we are able to derive explicit expressions for the three PN corrections  $\Phi_{PN}$ ,  $\alpha_{PN}$  and  $\beta_{PN}^i$  in terms of elliptic integrals.

Consider first the PN correction to the conformal factor. Substituting  $v^2 = \Omega^2 (x_1^2 + x_2^2)$  into the right hand of equation (58), the formal solution for  $\Phi_{PN}$  can be expressed as

$$\Phi_{PN}(\mathbf{x}) = \frac{5}{2}\rho_0 \int \frac{\Phi_N(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}' - \rho_0 \Omega^2 \int \frac{(x_1'^2 + x_2'^2)}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}'. \quad (75)$$

Inserting the analytical expression (73) for  $\Phi_N$  in the integrand of (75), it is easy to recognize that  $\Phi_{PN}$  can be written as

$$\Phi_{PN} = \frac{5}{2}\pi\rho_0 I_{Ch} \Phi_N + \frac{5}{2}\pi\rho_0 \sum_i A_i D_{ii} - \Omega^2 (D_{11} + D_{22}), \quad (76)$$

where the potential  $D_{ii}$ , written in terms of the index symbols  $A_{ijk\dots}$  and  $B_{ijk\dots}$ , is [see equation (3.132) in Ch69, Appendix D]

$$\begin{aligned} D_{ii} &= \rho_0 \int \frac{x_i'^2}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}' \\ &= \pi\rho_0 a_i^4 \left( A_{ii} - \sum_j A_{iij} x_j^2 \right) x_i^2 + \frac{1}{4}\pi\rho_0 a_i^2 \left( B_i - 2 \sum_j B_{ij} x_j^2 + \sum_{i,k} B_{ijk} x_j^2 x_k^2 \right). \end{aligned} \quad (77)$$

The derivation of  $\alpha_{PN}$  and  $\beta_i^{PN}$  is carried out in a similar way. The formal solution of equation (61) is

$$\Theta_{PN} = -\frac{3}{2}\rho_0\Omega^2 \int \frac{(x_1'^2 + x_2'^2)}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}' - 3 \int \frac{P(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}' + \frac{3}{4}\rho_0 \int \frac{\Phi_N(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}', \quad (78)$$

and, by substituting  $\Phi_N$  and the pressure profile (55), we obtain

$$\Theta_{PN} = 3\pi\rho_0 \left( A_3 a_3^2 + \frac{I_{Ch}}{4} \right) \Phi_N + \frac{15}{4}\pi\rho_0 \sum_i A_i D_{ii} - 3\Omega^2 (D_{11} + D_{22}). \quad (79)$$

Once  $\Theta_{PN}$  and  $\Phi_{PN}$  are known, the PN correction  $\alpha_{PN}$  follows trivially from (64).

Consider next the shift vector. Solving equations (70), (71) for  $G_{PN}^1$  and  $G_{PN}^2$  yields

$$G_{PN}^1 = 4\rho_0\Omega \int \frac{x_2'}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}', \quad G_{PN}^2 = -4\rho_0\Omega \int \frac{x_1'}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}', \quad (80)$$

hence

$$G_{PN}^1 = 4\Omega D_2, \quad G_{PN}^2 = -4\Omega D_1, \quad (81)$$

where [see equation (3.121) by Ch69]

$$\begin{aligned} D_i &= \rho_0 \int \frac{x_i'}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}' \\ &= \pi\rho_0 a_i^2 \left( A_i - \sum_j A_{ij} x_j^2 \right) x_i. \end{aligned} \quad (82)$$

Then, by differentiating expressions (82) with respect to  $x_i$ , we obtain the source term appearing in the differential equation (72) for  $B_{PN}$ . This gives

$$\nabla^2 B_{PN} = 8\pi\Omega\rho_0 A_{12} (a_1^2 - a_2^2) x_1 x_2, \quad (83)$$

where we used the symmetry property  $A_{12} = A_{21}$ . The solution is

$$\begin{aligned} B_{PN} &= -2\Omega\rho_0 A_{12} (a_1^2 - a_2^2) \int \frac{x_1' x_2'}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}' \\ &= -2\Omega A_{12} (a_1^2 - a_2^2) D_{12}, \end{aligned} \quad (84)$$

where

$$D_{12} = \pi\rho_0 a_1^2 a_2^2 \left( A_{12} - \sum_i A_{12i} x_i^2 \right) x_1 x_2. \quad (85)$$

Note that in axisymmetry, when  $a_1 = a_2$ ,  $B = 0$  and the shift vector is divergence-free [see equations (42), (44)]. To obtain the shift from (42), we must evaluate the gradient of the  $D_{12}$  according to

$$\nabla^i D_{12} = \frac{D_{12}}{x_i} (\delta_{i1} + \delta_{i2}) - 2\pi\rho_0 a_1^2 a_2^2 A_{12i} x_1 x_2 x_i, \quad (86)$$

where  $\delta_{ij}$  is the Kronecker delta. The final result is

$$\begin{aligned} \beta_{PN}^1 &= 4\Omega D_2 + \frac{1}{2}\Omega A_{12} \frac{a_1^2 - a_2^2}{x_1} (D_{12} - 2\pi\rho_0 a_1^2 a_2^2 x_1^3 x_2 A_{121}) \\ \beta_{PN}^2 &= -4\Omega D_1 + \frac{1}{2}\Omega A_{12} \frac{a_1^2 - a_2^2}{x_2} (D_{12} - 2\pi\rho_0 a_1^2 a_2^2 x_1 x_2^3 A_{122}) \\ \beta_{PN}^3 &= -\Omega A_{12} (a_1^2 - a_2^2) \pi\rho_0 a_1^2 a_2^2 x_1 x_2 x_3 A_{123}. \end{aligned} \quad (87)$$

Equations (73), (76), (79) and (87) completely determine the metric functions to PN order in the ellipsoidal approximation.<sup>4</sup> We compare this solution in Appendix B.1 to the exact solution for the interior metric in full general relativity for the case of a spherical, homogeneous configuration. We verify that our results agree with the PN expansion of the exact solution.

We have checked our results whenever possible with the corresponding PN expressions derived by CN, and they agree. A comparison is provided in Appendix A.

### 4.3. The Evaluation of the Conserved Quantities $M$ , $M_0$ and $J$

#### 4.3.1. The Integral Forms

With the metric coefficients derived in section (4.2) we can now derive explicit expressions for the total mass–energy  $M$ , the total rest–mass  $M_0$  and the angular momentum  $J$  of a homogeneous ellipsoid. These functionals will be then employed to determine the properties of the equilibrium configurations, using the energy variational method.

First consider the total baryon rest mass  $M_0$ , [CST92, equation (50)]

$$M_0 = \rho_0 \int_V U^t \sqrt{-g} d^3x, \quad (88)$$

where  $g = \det g_{\nu\mu}$  and  $V$  is the volume of the ellipsoid. In our adopted conformal gauge

$$M_0 = \rho_0 \int_V (\alpha U^t) \Psi^6 d^3x. \quad (89)$$

In order to evaluate the PN correction to  $M_0$ , we need to know the integrand in (89) up to order  $M^2/R^2$ . To this order the quantity  $\Psi^6$  can be approximated as

$$\begin{aligned} \Psi^6 &= \left(1 - \frac{\Phi}{2}\right)^6 \\ &\approx 1 - 3\Phi + \frac{15}{4}\Phi^2 \dots \\ &\approx 1 - 3\Phi_N - 3\Phi_{PN} + \frac{15}{4}\Phi_N^2 + O(M^3/R^3) \end{aligned} \quad (90)$$

In addition, we need a similar expansion for the quantity  $\alpha U^t$  and this can be derived from the normalization condition (50). After some algebra, we obtain

$$\begin{aligned} (\alpha U^t)^2 &\approx 1 + (U^t)^2 \Psi^4 [v^2 + 2(\beta_{PN}^2 v_2 + \beta_{PN}^1 v_1) + \dots] \\ &\approx 1 + (U^t)^2 (1 - 2\Phi_N + \dots) [v^2 + 2(\beta_{PN}^2 v_2 + \beta_{PN}^1 v_1) + \dots] \\ &\approx 1 + (U^t)^2 [v^2 - 2\Phi_N v^2 + 2(\beta_{PN}^2 v_2 + \beta_{PN}^1 v_1) + \dots]. \end{aligned} \quad (91)$$

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<sup>4</sup> In the first post–Newtonian approximation the orders of the metric coefficients that are needed are  $O(c^{-4})$ ,  $O(c^{-3})$  and  $O(c^{-2})$  for  $g_{00}$ ,  $g_{0\alpha}$  and  $g_{\alpha\beta}$  respectively. However, as reported in Table 1 in CN, to determine the total energy without any additional assumptions we need  $g_{\alpha\beta}$  to  $O(c^{-4})$ . An alternative method consists in evaluating the spatial metric  $g_{ij}$  to  $O(c^{-2})$ : in this case it is still possible to derive the conserved energy, provided that the equations are supplemented by a condition for isentropic flow (see Ch65a, Chandrasekhar 1969b, c for a detailed discussion about this point).



We can proceed via successive approximations, substituting in the right hand of (91) the expansion  $(U^t)^2 \approx 1 + (U^t)_N^2 + \dots$ , where  $(U^t)_N^2$  is the (unknown) Newtonian contribution to  $(U^t)^2$ . This yields

$$\begin{aligned} (\alpha U^t)^2 &\approx 1 + \left[1 + (U^t)_N^2 + \dots\right] \left[v^2 - 2\Phi_N v^2 + 2(\beta_{PN}^2 v_2 + \beta_{PN}^1 v_1) + \dots\right] \\ &\approx 1 + v^2 - 2\Phi_N v^2 + 2(\beta_{PN}^2 v_2 + \beta_{PN}^1 v_1) + (U^t)_N^2 v^2 + O\left(\frac{M^3}{R^3}\right). \end{aligned} \quad (92)$$

At Newtonian order we have

$$\begin{aligned} (U^t)^2 &\approx \alpha^{-2} (1 + v^2 \dots) \approx (1 - 2\Phi_N + \dots) (1 + v^2 \dots) \\ &\approx 1 - 2\Phi_N + v^2 + O\left(\frac{M^2}{R^2}\right) \end{aligned} \quad (93)$$

which gives  $(U^t)_N^2 = v^2 - 2\Phi_N$ , and

$$(\alpha U^t)^2 \approx 1 + v^2 - 4\Phi_N v^2 + 2(v_i \beta^i)_{PN} + v^4 + O\left(\frac{M^3}{R^3}\right), \quad (94)$$

where  $(v_i \beta^i)_{PN} = \beta_{PN}^1 v_1 + \beta_{PN}^2 v_2$ . We obtain

$$\alpha U^t \approx 1 + \frac{v^2}{2} - 2\Phi_N v^2 + (v_i \beta^i)_{PN} + \frac{3}{8} v^4 + O\left(\frac{M^3}{R^3}\right). \quad (95)$$

The two expansions (90) and (95) can be used together to yield

$$M_0 \approx \rho_0 \int_V \left[1 + \frac{v^2}{2} - 3\Phi_N - 3\Phi_{PN} + \frac{15}{4}\Phi_N^2 - \frac{7}{2}\Phi_N v^2 + \frac{3}{8}v^4 + (v_i \beta^i)_{PN}\right] d^3x, \quad (96)$$

and this expression can be written in a more compact form noting that

$$\int_V \Phi_{PN} d^3x = \int_V \left(v^2 \Phi_N - \frac{5}{2}\Phi_N^2\right) d^3x. \quad (97)$$

The latter result can be verified by substituting equations (73), (76) directly and integrating; the answer has been checked by making use of an algebraic manipulator (MAPLE). As a consequence, an equivalent expression for the baryon rest-mass is

$$M_0 \approx \rho_0 \int_V \left[1 + \frac{v^2}{2} - 3\Phi_N + \frac{45}{4}\Phi_N^2 - \frac{13}{2}\Phi_N v^2 + \frac{3}{8}v^4 + (v_i \beta^i)_{PN}\right] d^3x. \quad (98)$$

The second conserved quantity that enters in our calculation is the total mass-energy  $M$ , which we take as the ADM mass (see e.g. Bowen & York, 1980)

$$M = -\frac{1}{2\pi} \oint_{S_\infty} \nabla^i \Psi d^2 S_i = -\frac{1}{2\pi} \int_V \nabla^2 \Psi d^3x. \quad (99)$$

In equation (99) the surface integral is over the sphere at infinity, while Gauss's law has been used in the second step. Using the Hamiltonian constraint (35), the mass can be rewritten as

$$M = I_1 + I_2, \quad (100)$$

where

$$I_1 = \int \Psi^5 \rho d^3x \quad I_2 = \frac{1}{16\pi} \int \Psi^5 K^{ij} K_{ij} d^3x. \quad (101)$$

To evaluate (101) we note

$$\Psi^5 = \left(1 - \frac{\Phi}{2}\right)^5 \approx 1 - \frac{5}{2}\Phi_N - \frac{5}{2}\Phi_{PN} + \frac{5}{2}\Phi_N^2 + O\left(\frac{M^3}{R^3}\right), \quad (102)$$

while, by making use of expressions (39), (94), (34) we find

$$\rho \approx \rho_0 \left[ 1 + v^2 + v^4 - 4\Phi_N v^2 + 2(v_i \beta^i)_{PN} + \frac{P}{\rho_0} v^2 \right] + O\left(\frac{M^3}{R^3}\right), \quad (103)$$

$$\begin{aligned} K_{ij} K^{ij} &\approx \frac{1}{2} \left[ (\beta_{,2}^1)^2 + (\beta_{,1}^2)^2 + (\beta_{,3}^1)^2 + (\beta_{,1}^3)^2 + (\beta_{,3}^2)^2 + (\beta_{,2}^3)^2 + 2\beta_{,2}^1 \beta_{,1}^2 + 2\beta_{,3}^1 \beta_{,1}^3 + 2\beta_{,3}^2 \beta_{,2}^3 \right] \\ &+ \sum_{i=1}^3 (\beta_{,i}^i)^2 - \frac{1}{3} \left( \sum_{i=1}^3 \beta_{,i}^i \right)^2 + O\left(\frac{M^3}{R^3}\right), \end{aligned} \quad (104)$$

where  $y_{,i} \equiv \partial y / \partial x_i$ . For simplicity, the subscript “ $PN$ ” has been dropped and we simply write  $\beta^i$  for the leading term  $\beta_{PN}^i$ . The explicit form of (104) in terms of the index symbols is tedious but straightforward and is obtained by differentiating the shift function given by (87). The resulting expression has been computed with MAPLE. By collecting these quantities, exploiting (97) again and dropping from the integrand all terms of higher order, we obtain

$$\begin{aligned} M &\approx \rho_0 \int_V \left[ 1 + v^2 - \frac{5}{2}\Phi_N + \frac{35}{4}\Phi_N^2 - 9\Phi_N v^2 + \frac{P}{\rho_0} v^2 + v^4 + 2(v_i \beta^i)_{PN} \right] d^3x \\ &+ \frac{1}{16\pi} \int_V K_{ij} K^{ij} d^3x. \end{aligned} \quad (105)$$

For later applications it is also useful to write down the conserved energy  $E = M - M_0$

$$\begin{aligned} M - M_0 &\approx \rho_0 \int_V \left[ \frac{1}{2}\Phi_N + \frac{1}{2}v^2 - \frac{5}{2}\Phi_N^2 - \frac{5}{2}\Phi_N v^2 + \frac{P}{\rho_0} v^2 + \frac{5}{8}v^4 + (v_i \beta^i)_{PN} \right] d^3x \\ &+ \frac{1}{16\pi} \int_V K_{ij} K^{ij} d^3x. \end{aligned} \quad (106)$$

Finally, we need to evaluate the total angular momentum,  $J$ . This quantity is obtained from the integral (see e.g. Bowen & York, 1980)

$$J_i = \frac{1}{16\pi} \epsilon_{ijk} \oint_{S_\infty} (x^j K^{km} - x^k K^{jm}) d^2 S_m, \quad (107)$$

where  $\epsilon_{ijk}$  is the Levi-Civita tensor and integration is over the sphere at infinity. Let us also introduce the symmetric tensor  $\hat{K}^{ij} = \Psi^2 K^{ij}$ . Since it is  $\Psi \rightarrow 1 + O(r^{-1})$  as  $r \rightarrow \infty$ ,  $\hat{K}^{ij}$  may be used in place of the extrinsic curvature in (107), to lowest order. Moreover, again following Bowen & York (1980), this tensor satisfies

$$\nabla_l \hat{K}^{kl} = \Psi^{10} D_l K^{kl}. \quad (108)$$

It follows that

$$\begin{aligned}
J_i &= \frac{1}{16\pi} \epsilon_{ijk} \oint_{S_\infty} \left( x^j \hat{K}^{km} - x^k \hat{K}^{jm} \right) d^2 S_m \\
&= \frac{1}{8\pi} \epsilon_{ijk} \oint_{S_\infty} x^j \hat{K}^{km} d^2 S_m \\
&= \frac{1}{8\pi} \epsilon_{ijk} \int x^j \nabla_l \hat{K}^{kl} d^3 x,
\end{aligned} \tag{109}$$

and, using (108) and the momentum constraint (37), this can be rewritten as

$$J_i = \epsilon_{ijk} \int x^j S^k \Psi^{10} d^3 x. \tag{110}$$

For rotation about the  $x_3$  axis, we have  $J_3 = J$  and the total angular momentum turns out to be

$$\begin{aligned}
J &= \epsilon_{3jk} \int x^j S^k \Psi^{10} d^3 x \\
&= \int \Psi^{10} (x^1 S^2 - x^2 S^1) d^3 x,
\end{aligned} \tag{111}$$

Exploiting equations (41), (52), (68) and (93), we are able to derive the PN approximation of the integrand in (111):

$$\begin{aligned}
S^i &= \rho_0 \left( 1 + \frac{P}{\rho_0} \right) (\alpha U^t) \gamma^{ij} U_j \\
&\approx \rho_0 \left( 1 + \frac{P}{\rho_0} \right) \left( 1 + \frac{1}{2} v^2 \dots \right) \left( 1 + \frac{1}{2} v^2 - \Phi_N + \dots \right) (\beta_{PN}^i + v^i + \dots) \\
&\approx \rho_0 v^i \left( 1 - \Phi_N + v^2 + \frac{P}{\rho_0} \right) + \rho_0 \beta_{PN}^i,
\end{aligned} \tag{112}$$

and

$$\begin{aligned}
\Psi^{10} x^j S^i &\approx \left( 1 - \frac{\Phi_N}{2} + \dots \right)^{10} x^j S^i \\
&\approx x^j \rho_0 \left[ v^i \left( 1 - 6\Phi_N + v^2 + \frac{P}{\rho_0} \right) + \beta_{PN}^i \right].
\end{aligned} \tag{113}$$

Substituting (113) into (111), we find

$$J \approx \frac{\rho_0}{\Omega} \int_V \left[ v^2 - 6v^2 \Phi_N + v^4 + \frac{P}{\rho_0} v^2 + (v_i \beta^i)_{PN} \right] d^3 x. \tag{114}$$

where all terms of order  $O(M^3/R^3)$  have been neglected in the integrand. The kinetic energy is  $T = \Omega J/2$  [see CST92, equation (56)], and it is easy to recognize that (114) is correct at the PN level.

The derivation of conservation laws in general relativity has been considered by Chandrasekhar (1969b) and CN, who used the symmetric energy–momentum complex of Landau–Lisfhitz to determine integral forms of conserved quantities at various post–Newtonian orders. Although CN do not present any expressions for the integrated conserved quantities, their results provide a useful tool to check the correctness of (106) and (114) by comparing the corresponding integrands. This comparison is presented in Appendix A.3.

### 4.3.2. Evaluating The Conserved Integrals

Summarizing, integral expressions for the conserved quantities that we have derived are

$$\begin{aligned}
M &\approx \rho_0 \int_V \left[ 1 + v^2 - \frac{5}{2} \Phi_N + \frac{35}{4} \Phi_N^2 - 9 \Phi_N v^2 + \frac{P}{\rho_0} v^2 + v^4 + 2 (v_i \beta^i)_{PN} \right] d^3 x \\
&+ \frac{1}{16\pi} \int_V K_{ij} K^{ij} d^3 x, \\
M_0 &\approx \rho_0 \int_V \left[ 1 + \frac{v^2}{2} - 3 \Phi_N - 3 \Phi_{PN} + \frac{15}{4} \Phi_N^2 - \frac{7}{2} \Phi_N v^2 + \frac{3}{8} v^4 + (v_i \beta^i)_{PN} \right] d^3 x, \\
M - M_0 &\approx \rho_0 \int_V \left[ \frac{1}{2} \Phi_N + \frac{1}{2} v^2 - \frac{5}{2} \Phi_N^2 - \frac{5}{2} \Phi_N v^2 + \frac{P}{\rho_0} v^2 + \frac{5}{8} v^4 + (v_i \beta^i)_{PN} \right] d^3 x \\
&+ \frac{1}{16\pi} \int_V K_{ij} K^{ij} d^3 x, \\
J &\approx \frac{\rho_0}{\Omega} \int_V \left[ v^2 - 6 v^2 \Phi_N + v^4 + \frac{P}{\rho_0} v^2 + (v_i \beta^i)_{PN} \right] d^3 x,
\end{aligned} \tag{115}$$

In order to perform the quadratures over the fluid volume, we adopt the ellipsoidal approximation for the matter distribution, whereas  $\rho_0 = \text{constant}$  inside a triaxial ellipsoid. With this assumption, it is convenient to introduce and evaluate the following integrals:

$$\mathcal{I}_1 = \int_V \rho_0 d^3 x \equiv M_c = \frac{4\pi}{3} \rho_0 a_1 a_2 a_3 = \frac{4\pi}{3} \rho_0 R^3, \tag{116}$$

$$\mathcal{I}_2 = \int_V \rho_0 \Phi_N d^3 x = 2\mathcal{M} \equiv 2 \sum_i \mathcal{M}_{ii} = -\frac{6}{5} \frac{M_c^2}{R} f, \tag{117}$$

$$\mathcal{I}_3 = \int_V \rho_0 v^2 d^3 x = \Omega^2 \int_V \rho_0 (x_1^2 + x_2^2) d^3 x = \Omega^2 (I_{11} + I_{22}) = \frac{2}{5} M_c \Omega^2 R^2 \frac{1}{h}, \tag{118}$$

where  $\mathcal{M}$  is the Newtonian potential energy, while  $R$  and the two dimensionless ratios  $f$  and  $h$  have been introduced in section (3) [see equations (7), (10), (11)]. In deriving the previous expressions we made use of the results

$$\mathcal{M} = \sum_i \mathcal{M}_{ii}, \tag{119}$$

$$\mathcal{M}_{ii} = -2\pi \rho_0 A_i I_{ii}, \tag{120}$$

$$I_{ij} = \int_V \rho_0 x_i x_j d^3 x = \frac{1}{5} M_c a_i^2 \delta_{ij}, \tag{121}$$

given by expression (2.4), (2.12), (2.13), (3.128), (3.129) in Ch69. Note that, in contrast to the Newtonian analysis,  $M_c$  is only a coordinate quantity and has no physical meaning. The other integrals appearing in equations (115) are

$$\begin{aligned}
\mathcal{I}_4 &= \int_V \rho_0 \Phi_{PN} d^3 x, & \mathcal{I}_5 &= \int_V \rho_0 \Phi_N^2 d^3 x, \\
\mathcal{I}_6 &= \int_V P v^2 d^3 x, & \mathcal{I}_7 &= \int_V \rho_0 \Phi_N v^2 d^3 x, \\
\mathcal{I}_8 &= \int_V \rho_0 v^4 d^3 x, & \mathcal{I}_9 &= \int_V \rho_0 \Omega (x_1 \beta_{PN}^2 - \beta_{PN}^1 x_2) d^3 x, \\
\mathcal{I}_{10} &= \frac{1}{16\pi} \int_V K^{ij} K_{ij} d^3 x.
\end{aligned} \tag{122}$$

From the quadratic forms of  $\Phi_N$ ,  $\Phi_{PN}$ , etc... it is easy to recognize that the evaluation of  $\mathcal{I}_i$ ,  $i = 4..9$ , only involves quadratures of the kind  $I_{ij}$  and  $E_{ij}$ , where<sup>5</sup>

$$E_{ij} = \int_V \rho_0 x_i^2 x_j^2 d^3x = \frac{1}{35} M_c a_i^2 a_j^2 (1 + 2\delta_{ij}) ; \quad (123)$$

the same finding holds also in the case of  $I_{10}$ , but the integration involves some lengthy algebraic calculations. After many simplifications, and exploiting the properties of the index symbols, we obtain with the help of MAPLE the following integrated forms of the conserved quantities

$$M \approx M_c + 3 \frac{M_c^2}{R} f + \frac{2}{5} M_c \Omega^2 R^2 \frac{1}{h} + \frac{M_c^3}{R^2} g_1 + \frac{M_c^2}{R} \Omega^2 R^2 p_1 , \quad (124)$$

$$M_0 \approx M_c + \frac{18}{5} \frac{M_c^2}{R} f + \frac{1}{5} M_c \Omega^2 R^2 \frac{1}{h} + \frac{M_c^3}{R^2} g_2 + \frac{M_c^2}{R} \Omega^2 R^2 p_2 , \quad (125)$$

$$J \approx \Omega M_c R^2 \frac{2}{5h} \left( 1 + \frac{5}{2} \frac{M_c}{R} p_3 h \right) , \quad (126)$$

$$M - M_0 \approx -\frac{3}{5} \frac{M_c^2}{R} f + \frac{1}{5} M_c \Omega^2 R^2 \frac{1}{h} + \frac{M_c^3}{R^2} g_{12} + \frac{M_c^2}{R} \Omega^2 R^2 p_{12} , \quad (127)$$

where the functions  $g_i$ ,  $p_i$  are defined as (see Appendix D)

$$g_1 = \frac{99}{8} f^2 + \frac{9}{32} \frac{1}{\lambda_1 \lambda_2} \sum_l A_l^2 \frac{a_l^4}{a_1^2 a_2^2} , \quad (128)$$

$$g_2 = \frac{891}{56} f^2 + \frac{81}{224} \frac{1}{\lambda_1 \lambda_2} \sum_l A_l^2 \frac{a_l^4}{a_1^2 a_2^2} , \quad (129)$$

$$g_{12} \equiv g_1 - g_2 = -\frac{99}{28} f^2 - \frac{9}{112} \frac{1}{\lambda_1 \lambda_2} \sum_l A_l^2 \frac{a_l^4}{a_1^2 a_2^2} , \quad (130)$$

$$p_1 = \frac{6}{7} \frac{f}{h} + \frac{57}{35} A_3 \frac{\lambda_1 \lambda_2}{h} + \frac{69}{70} \frac{A_1 + A_2}{\lambda_1 \lambda_2} + \frac{3}{70} \frac{1}{\lambda_1 \lambda_2} (A_1 - A_2)^2 + \tilde{\mathcal{I}}_{10} , \quad (131)$$

$$p_2 = \frac{21}{20} \frac{f}{h} + \frac{57}{56} A_3 \frac{\lambda_1 \lambda_2}{h} + \frac{33}{56} \frac{A_1 + A_2}{\lambda_1 \lambda_2} + \frac{3}{140} \frac{1}{\lambda_1 \lambda_2} (A_1 - A_2)^2 , \quad (132)$$

$$p_{12} \equiv p_1 - p_2 = -\frac{27}{140} \frac{f}{h} + \frac{171}{280} A_3 \frac{\lambda_1 \lambda_2}{h} + \frac{111}{280} \frac{A_1 + A_2}{\lambda_1 \lambda_2} + \frac{3}{140} \frac{1}{\lambda_1 \lambda_2} (A_1 - A_2)^2 + \tilde{\mathcal{I}}_{10} , \quad (133)$$

$$p_3 = \frac{6}{5} \frac{f}{h} + \frac{24}{35} A_3 \frac{\lambda_1 \lambda_2}{h} + \frac{18}{35} \frac{A_1 + A_2}{\lambda_1 \lambda_2} + \frac{3}{140} \frac{1}{\lambda_1 \lambda_2} (A_1 - A_2)^2 . \quad (134)$$

Due to the complexity of the  $K^{ij} K_{ij}$  contribution appearing into  $M$ , we simplify the notation by leaving the term  $\tilde{\mathcal{I}}_{10} = \left( \frac{M_c^2}{R} \Omega^2 R^2 \right)^{-1} \mathcal{I}_{10}$ , without substituting the explicit form in terms of the ellipsoidal variables.

For an expression in terms of these variables, see Appendix C. The structure of expressions (124)–(127) is particularly convenient for performing the required variations, since the full dependence on the two axial ratios is contained in  $f$ ,  $h$  and, for the PN contributions, in  $g_i$  and  $p_i$ . We have checked that our result agrees with the PN correction to the Newtonian energy obtained by ST for homogeneous spheres (Appendix B.2).

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<sup>5</sup>The integrals  $E_{ij}$  are easily evaluated by introducing the variables  $x'_i = x_i/a_i$ , such that the equation of the ellipsoid is  $\sum_i (x'_i)^2 = 1$ , and using spherical polar coordinates in the new system.

## 5. Equilibrium Configurations

As we did in the Newtonian case (see section 3), we now construct sequences of axisymmetric equilibrium models. Each sequence can be parametrized by, e.g., the value  $M/R_s$  of the spherical, nonrotating member. These configurations yield a PN generalization of Maclaurin spheroids and were originally investigated by Chandrasekhar (1965b) by using the tensor virial method. As we will confirm, the effect of general relativity is to attribute to a star of a given eccentricity a larger value of  $\Omega^2/(\pi\rho_0)$ . Sequences of relativistic, numerical models at fixed  $M_0\sqrt{\rho_0}$  have been also published by BI (see also Bonazzola & Schneider 1974).

We construct the equilibrium sequence by minimizing  $M$  or, equivalently,  $M - M_0$ , keeping fixed  $M_0$  and  $J$ . Due to the complexity of the expressions, part of the calculations here and in the next section have been done by using an algebraic manipulator (MAPLE). The procedure is as follows. First, we combine expressions (126) and (127) and rewrite the quantity  $M - M_0$  as a function of  $R$ ,  $\lambda_1$ ,  $\lambda_2$ ,  $\rho_0$  and  $J$ . This gives

$$M - M_0 \approx -\frac{3}{5} \frac{M_c^2}{R} f + \frac{5}{4} \frac{J^2}{M_c R^2} h + \frac{M_c^3}{R^2} g_{12} + \frac{25}{4} \frac{J^2}{R^3} h^2 p_{123}, \quad (135)$$

where  $M_c = M_c(\rho_0, R)$ ,  $p_{123} = p_{12} - p_3$  and where we used the relation

$$\begin{aligned} \Omega^2 R^2 &\approx \frac{J^2}{M_c^2 R^2} \frac{25h^2}{4} \left(1 + \frac{5}{2} \frac{M_c}{R} p_3 h\right)^{-2} \\ &\approx \frac{J^2}{M_c^2 R^2} \frac{25h^2}{4} \left(1 - 5 \frac{M_c}{R} p_3 h\right). \end{aligned} \quad (136)$$

The equilibrium sequence is then determined by minimizing  $M - M_0$  according to

$$\frac{\partial(M - M_0)}{\partial\lambda_i} = \frac{\partial(M - M_0)}{\partial R} \frac{\partial R}{\partial\lambda_i} + \left[ \frac{\partial(M - M_0)}{\partial\lambda_i} \right]_{R=const} = 0, \quad (137)$$

where, here and in the following, the partial derivative with respect to  $\lambda_i$  is taken holding constant  $J$ ,  $\rho_0$  and  $\lambda_j$ , with  $i \neq j$

$$\frac{\partial}{\partial\lambda_i} \equiv \left( \frac{\partial}{\partial\lambda_i} \right)_{J, \rho_0, \lambda_j (i \neq j)}. \quad (138)$$

Note that, in contrast to the Newtonian case, the mean radius  $R$  is no longer constant. The variation of  $R$  can be obtained exploiting the constraint  $dM_0 = 0$ , which gives

$$\frac{\partial R}{\partial\lambda_i} = - \left( \frac{\partial M_0}{\partial R} \right)^{-1} \left( \frac{\partial M_0}{\partial\lambda_i} \right)_{R=const}. \quad (139)$$

Since

$$M_0 \approx M_c + \frac{18}{5} \frac{M_c}{R^2} f + \frac{5}{4} \frac{J^2}{M_c R^2} h + \dots, \quad (140)$$

expression (139) can be approximated as

$$\frac{\partial R}{\partial\lambda_i} \approx M_c \left[ -\frac{6}{5} \left( \frac{\partial f}{\partial\lambda_i} \right)_{R=const} - \frac{5}{12} \frac{J^2}{M_c^3 R} \left( \frac{\partial h}{\partial\lambda_i} \right)_{R=const} \right]. \quad (141)$$

In axisymmetry, we set  $a_1 = a_2$  when performing the two variations (137). Since, in this case, the first derivatives with respect to the two axial ratios are equal, we need only consider the case  $i = 1$ . To simplify

the notation, we introduce the following symbols for the derivatives of a given function  $X$

$$\begin{aligned} X^{(1)} &\equiv \left( \frac{\partial X}{\partial \lambda_1} \right)_{R=\text{const}}, & X^{(2)} &\equiv \left( \frac{\partial X}{\partial \lambda_2} \right)_{R=\text{const}}, \\ X^{(12)} &\equiv \left( \frac{\partial^2 X}{\partial \lambda_1 \partial \lambda_2} \right)_{R=\text{const}}, & X^{(11)} &\equiv \left( \frac{\partial^2 X}{\partial \lambda_1^2} \right)_{R=\text{const}}. \end{aligned}$$

At the PN order, the variation (137) with respect to  $\lambda_1$  is

$$\left( -3 \frac{M_c^2}{R^2} f - \frac{25}{4} \frac{J^2}{M_c R^3} h + \dots \right) \frac{\partial R}{\partial \lambda_1} - \frac{3}{5} \frac{M_c^2}{R} f^{(1)} + \frac{5}{4} \frac{J^2}{M_c R^2} h^{(1)} + \frac{M_c^3}{R^2} g_{12}^{(1)} + \frac{25}{4} \frac{J^2}{R^3} (h^2 p_{123})^{(1)} = 0, \quad (142)$$

and this expression can be combined with (141) to yield

$$\begin{aligned} -\frac{3}{5} f^{(1)} + \frac{5}{4} \frac{J^2}{M_c^3 R} h^{(1)} &+ \frac{M_c}{R} \left[ g_{12}^{(1)} + \frac{25}{4} \frac{J^2}{M_c^3 R} (h^2 p_{123})^{(1)} + \frac{18}{5} f f^{(1)} \right. \\ &\left. + \frac{5}{4} \frac{J^2}{M_c^3 R} f h^{(1)} + \frac{15}{2} \frac{J^2}{M_c^3 R} h f^{(1)} + \frac{125}{48} \frac{J^4}{M_c^6 R^2} h h^{(1)} \right] = 0. \end{aligned} \quad (143)$$

It is now more convenient to reexpress the previous equation in terms of the gauge invariant parameter  $\Omega^2 / (\pi \rho_0)$ . From equation (126) we obtain

$$\frac{J^2}{M_c^3 R} \approx \frac{\Omega^2}{\pi \rho_0} \frac{3}{25 h^2} \left( 1 + 5 \frac{M_c}{R} p_3 h \right). \quad (144)$$

By using this result, the equilibrium condition (143) becomes

$$\begin{aligned} \frac{\Omega^2}{\pi \rho_0} = \frac{4 h^2 f^{(1)}}{h^{(1)}} &- \frac{20}{3} \frac{h^2}{h^{(1)}} \frac{M_c}{R} \left\{ g_{12}^{(1)} + \frac{18}{5} f f^{(1)} \right. \\ &\left. + \frac{\Omega^2}{\pi \rho_0} \left[ \frac{3}{4 h^2} (h^2 p_{123})^{(1)} + \frac{3}{4} \frac{h^{(1)}}{h} p_3 + \frac{3}{20} \frac{f h^{(1)}}{h^2} + \frac{21}{20} \frac{f^{(1)}}{h} \right] \right\}, \end{aligned} \quad (145)$$

where, in the right hand side, the first term represents the Newtonian result, while the remaining terms give the PN correction. Finally we set  $a_1 = a_2$ , using expressions (D13), (D14). The derivatives of  $f$  and  $h$  are derived using expressions (A3), (A9) in LRS, while for the derivatives of  $g_{12}$  and  $p_{123}$  we used results reported in Appendix D.2 (expressions D16). Doing this, we obtain the relation between the angular velocity and the eccentricity along the relativistic equilibrium sequence. By adopting the same notation as in Chandrasekhar (1965b), we write the resulting expression in the form

$$\frac{\Omega^2}{\pi \rho_0} = 8 \lambda^2 f^{(1)} \Big|_{a_1=a_2} + \frac{2 M_c}{a_1} E(e), \quad (146)$$

where  $\lambda \equiv \lambda_1 = \lambda_2 = (1 - e^2)^{1/3}$  and where the strength of relativity is measured by the compaction parameter  $2 M_c / a_1$ ,

$$\begin{aligned} E(e) &= -\frac{20}{3} \lambda^{3/2} \left\{ g_{12}^{(1)} + \frac{18}{5} f f^{(1)} \right. \\ &\left. + \frac{\Omega^2}{\pi \rho_0} \left[ \frac{3}{4 \lambda^2} (h^2 p_{123})^{(1)} + \frac{3}{8} \frac{p_3}{\lambda} + \frac{3}{40} \frac{f}{\lambda^2} + \frac{21}{20} \frac{f^{(1)}}{\lambda} \right] \right\} \Big|_{a_1=a_2}, \end{aligned} \quad (147)$$

is a function only of the eccentricity of the spheroid, and

$$8\lambda^2 f^{(1)} \Big|_{a_1=a_2} = 2 \left( A_1 - \frac{a_3^2}{a_1^2} A_3 \right), \quad (148)$$

which formally coincides with the Newtonian expression (25). However, we emphasize that the eccentricity which enters in our PN formalism (and that of Chandrasekhar 1965b) is defined in terms of the ratio of the coordinate quantities  $a_3$  and  $a_1$  and is not a gauge invariant parameter.

The PN sequences of equilibrium are reported in Figure 1 for different values of the parameter  $M/R_s$ , where  $R_s$  is the equatorial radius in Schwarzschild coordinates, and, in the spherical limit

$$\frac{M_c}{a_1} = \frac{M}{R_s} \frac{1}{4} \left( 1 - \frac{M}{R_s} + \sqrt{1 - \frac{2M}{R_s}} \right)^2, \quad (149)$$

[see Lightman *et al.* 1979, expression (4), page 422]. Note that this function has a maximum for  $M/R_s = (M/R_s)_{max} = 5/18$ , corresponding to  $2M_c/a_1 = 3125/11664 \approx 0.268$ . It follows that our PN formalism can be used to investigate relativistic sequences up to a maximum value  $(M/R_s)_{max} = 5/18 \approx 0.28$ . As we can see from Figure 1, we find that in the relativistic case the value of  $\Omega^2/(\pi\rho_0)$  is larger than what Newtonian theory predicts for the same value of the eccentricity, confirming the Chandrasekhar’s (1965b) results. Sequences of equilibrium can be described in an equivalent way by using, in place of  $\Omega^2/(\pi\rho_0)$ , the ratio

$$\frac{T}{|W|} \equiv \frac{\frac{1}{2}\Omega J}{\left(\frac{1}{2}\Omega J + M_0 - M\right)}, \quad (150)$$

which reduces to the ratio of rotational kinetic energy to gravitational potential energy in the Newtonian limit. Note that  $T/|W|$  is gauge invariant for rigidly rotating objects, since  $M - M_0$ ,  $J$  and  $\Omega$  are gauge invariant (they can all be measured by observers at large distance from the spheroid). By using the expressions of  $J$ ,  $M - M_0$  and setting  $a_1 = a_2$ , we find, to PN order,

$$\frac{T}{|W|} = \frac{\Omega^2}{\pi\rho_0} \frac{1}{2(2A_1 + A_3\lambda^3)} \left\{ 1 + \frac{2M_c}{a_1} \frac{5}{2\sqrt{\lambda}} \left[ \frac{p_3 h}{2} + \frac{g_{12}}{3f} + \frac{1}{4f} \frac{\Omega^2}{\pi\rho_0} \left( p_{12} - \frac{1}{2} p_3 \right) \right] \right\}_{\lambda_1=\lambda_2=\lambda}. \quad (151)$$

Figure 2 shows the sequence of equilibrium corresponding to the maximum value  $(M/R_s)_{max} = 5/18$  together with the Newtonian Maclaurin sequence ( $M/R_s = 0$ ). As we can see, in the range we are considering,  $T/|W|$  is insensitive to the parameter  $M/R_s$ . Figure 3 shows the relation between the two gauge invariant quantities  $\Omega^2/\pi\rho_0$  and  $T/|W|$ , for the same equilibrium sequences as in Figure 1; squares mark the secular instability point and will be discussed in the next section.

The function  $E(e)$  and the corresponding function  $E_{Ch}$  obtained by Chandrasekhar (1965b) are compared in Table 1 and Figure 4. As it can be seen, the two expressions are in close agreement up at least to the value of  $e$  corresponding to the secular instability point in Newtonian theory, with a maximum fractional difference of a few percent. The discrepancy increases at larger eccentricity (30 % at the Newtonian dynamical instability point). This discrepancy may be ascribed mainly to our coordinate ellipsoidal approximation for the deformation and equilibrium shape. This assumption is only exact in the Newtonian limit. For  $e \ll 1$ , the two functions can be expanded as

$$E \approx 0.219e^2 + 0.115e^4 + 0.059e^6 + \dots, \quad (152)$$

$$E_{Ch} \approx 0.228e^2 + 0.114e^4 + 0.042e^6 + \dots. \quad (153)$$



We have compared our results with the fully relativistic, numerical sequences presented in Figure 1 in the paper by BI. To compare those computations with our PN models, we have restricted our attention to the least relativistic sequence in BI, which corresponds to  $2M_c/a_1 = 0.206$  ( $\gamma_s = 0.154$  in BI). To make the comparison, we note that BI parametrize in terms of a proper eccentricity, which is defined as

$$e_{BI}^2 = 1 - \frac{d_p^2}{d_e^2}, \quad (154)$$

$$d_p = \int_0^{a_3} \Psi^2|_{x_1=x_2=0} dx_3, \quad d_e = \int_0^{a_1} \Psi^2|_{x_2=x_3=0} dx_1, \quad (155)$$

where  $d_p$  and  $d_e$  are the proper radii in the polar and equatorial directions, respectively. Using (56) and (73) for  $\Psi$ , we find

$$\begin{aligned} e_{BI}^2 &\approx e^2 + \frac{2M_c}{a_1} \sqrt{1-e^2} \left[ \frac{A_1}{2} \left( \frac{3}{2} - e^2 \right) - \frac{1-e^2}{2} \right] \\ &\approx e^2 + \frac{2M_c}{a_1} \left( \frac{1}{15}e^2 - \frac{1}{42}e^4 - \frac{11}{840}e^6 + \dots \right), \end{aligned} \quad (156)$$

which provides the relation between the two definitions of eccentricity.

Figure 5 shows the comparison between the BI values of  $\Omega^2/(\pi\rho_0)$  and the two PN sequences derived by using  $E(e)$  and the Chandrasekhar (1965b) expression  $E_{Ch}$ , with  $2M_c/a_1 = 0.206$ ; Table 2 contains the values of  $\Omega^2/(\pi\rho_0)$ , evaluated by using our expression, and the corresponding quantities by BI. In order to make a comparison between the relativistic corrections, which is more useful, we define the correction as the difference between the relativistic and the Newtonian value of  $\Omega^2/(\pi\rho_0)$  [given by expression (148)] and compare these corrections. Since BI does not present tabulated values for  $\Omega^2/(\pi\rho_0)$ , the values we have reported in table 2 have been read off Figure 1 in BI, with an estimated error of  $\sim 0.02$ . For  $e < 0.7$ , this error is of the same order as the PN correction, making the measured value of the latter questionable. For this reason we have reported in Table 2 only the values of this quantity for  $e$  higher than 0.7, together with the corresponding fractional error. As it can be seen, in this case our values are lower with respect to the numerical ones by BI, but the difference in the corrections is at most  $\approx 10\%$  for  $e \approx 0.8$ .

## 6. The Secular Instability Point

The determination of the point of onset of secular instability proceeds as in the Newtonian case [cf. equations (26), (28)], whereby we must evaluate

$$\left. \frac{\partial^2 (M - M_0)}{\partial \lambda_1^2} \right|_{\lambda_1=\lambda_2=\lambda} = \left. \frac{\partial^2 (M - M_0)}{\partial \lambda_2 \partial \lambda_1} \right|_{\lambda_1=\lambda_2=\lambda}. \quad (157)$$

By using expression (137), the second derivatives can be written as

$$\frac{\partial^2 (M - M_0)}{\partial \lambda_1^2} = \frac{\partial}{\partial \lambda_1} \left\{ \frac{\partial (M - M_0)}{\partial R} \frac{\partial R}{\partial \lambda_1} + \left[ \frac{\partial (M - M_0)}{\partial \lambda_i} \right]_{R=const} \right\}, \quad (158)$$

$$\frac{\partial^2 (M - M_0)}{\partial \lambda_2 \partial \lambda_1} = \frac{\partial}{\partial \lambda_2} \left\{ \frac{\partial (M - M_0)}{\partial R} \frac{\partial R}{\partial \lambda_1} + \left[ \frac{\partial (M - M_0)}{\partial \lambda_i} \right]_{R=const} \right\}, \quad (159)$$

where the expression in brackets is given by equation (143), multiplied by  $M_c^2/R$ . After some algebra we find that, at PN order,

$$\begin{aligned} \frac{\partial^2 (M - M_0)}{\partial \lambda_1^2} &= \frac{M_c^2}{R} \left[ -\frac{3}{5} f^{(11)} + \frac{5}{4} \frac{J^2}{M_c^3 R} h^{(11)} \right] + \frac{M_c^3}{R^2} \left[ \frac{36}{5} \left( f^{(1)} \right)^2 + \frac{18}{5} f f^{(11)} + \frac{35}{2} \frac{J^2}{M_c^3 R} f^{(1)} h^{(1)} \right. \\ &+ \frac{5}{4} \frac{J^2}{M_c^3 R} f h^{(11)} + \frac{15}{2} \frac{J^2}{M_c^3 R} h f^{(11)} + \frac{125}{24} \frac{J^4}{M_c^6 R^2} \left( h^{(1)} \right)^2 + \frac{125}{48} \frac{J^4}{M_c^6 R^2} h h^{(11)} \\ &\left. + g_{12}^{(11)} + \frac{25}{4} \frac{J^2}{M_c^3 R} (h^2 p_{123})^{(11)} \right], \end{aligned} \quad (160)$$

$$\begin{aligned} \frac{\partial^2 (M - M_0)}{\partial \lambda_1 \partial \lambda_2} &= \frac{M_c^2}{R} \left[ -\frac{3}{5} f^{(12)} + \frac{5}{4} \frac{J^2}{M_c^3 R} h^{(12)} \right] + \frac{M_c^3}{R^2} \left[ \frac{36}{5} f^{(2)} f^{(1)} + \frac{18}{5} f f^{(12)} + \frac{35}{4} \frac{J^2}{M_c^3 R} \left( f^{(1)} h^{(2)} \right. \right. \\ &+ \left. \left. f^{(2)} h^{(1)} \right) + \frac{5}{4} \frac{J^2}{M_c^3 R} f h^{(12)} + \frac{15}{2} \frac{J^2}{M_c^3 R} h f^{(12)} + \frac{125}{24} \frac{J^4}{M_c^6 R^2} h^{(1)} h^{(2)} + \frac{125}{48} \frac{J^4}{M_c^6 R^2} h h^{(12)} \right. \\ &\left. + g_{12}^{(12)} + \frac{25}{4} \frac{J^2}{M_c^3 R} (h^2 p_{123})^{(12)} \right]. \end{aligned} \quad (161)$$

Given axisymmetry, we can set  $a_1 = a_2$  when equating expressions (160) and (161); the equivalence between the subscripts 1 and 2 will be made explicit at the end of the calculations. After this substitution is made all terms containing first derivatives cancel in the final relation; hence, for the sake of simplicity, they will be dropped in the following expressions. Imposing condition (157) yields

$$\begin{aligned} \frac{5}{4} \frac{J^2}{M_c^3 R} \mathcal{D}[h] &= \frac{3}{5} \mathcal{D}[f] - \frac{M_c}{R} \left\{ \frac{18}{5} f \mathcal{D}[f] + \frac{5}{4} \frac{J^2}{M_c^3 R} f \mathcal{D}[h] + \frac{15}{2} \frac{J^2}{M_c^3 R} h \mathcal{D}[f] \right. \\ &+ \left. \frac{125}{48} \frac{J^4}{M_c^6 R^2} h \mathcal{D}[h] + \mathcal{D}[g_{12}] + \frac{25}{4} \frac{J^2}{M_c^3 R} \mathcal{D}[h^2 p_{123}] \right\}, \end{aligned} \quad (162)$$

where the difference operator  $\mathcal{D}$  is defined as the difference between the two second derivatives of a function with respect to the axial ratios

$$\mathcal{D}[X] \equiv X^{(12)} - X^{(11)}. \quad (163)$$

Replacing  $J^2$  with  $\Omega^2$  according to (144), expression (162) becomes

$$\begin{aligned} \frac{\Omega^2}{\pi \rho_0} &= 4h^2 \frac{\mathcal{D}[f]}{\mathcal{D}[h]} + \frac{20}{3} \frac{M_c}{R} \frac{h^2}{\mathcal{D}[h]} \left\{ \mathcal{D}[g_{12}] + \frac{21}{5} f \mathcal{D}[f] \right. \\ &+ \left. 3p_3 h \mathcal{D}[f] + \frac{\Omega^2}{\pi \rho_0} \left[ \frac{21}{20h} \mathcal{D}[f] + \frac{3}{4h^2} \mathcal{D}[h^2 p_{123}] \right] \right\}. \end{aligned} \quad (164)$$

Setting now  $\lambda_1 = \lambda_2 = \lambda$  gives

$$\frac{\Omega^2}{\pi \rho_0} = \frac{1}{5} (9B_{11} + A_1 - \lambda^3 A_3) + \frac{2M_c}{a_1} \frac{9}{10} P(e), \quad (165)$$

where

$$\begin{aligned} P(e) &= -\frac{20}{27} \lambda^{5/2} \left\{ \mathcal{D}[g_{12}] + \frac{21}{5} f \mathcal{D}[f] \right. \\ &+ \left. 3p_3 h \mathcal{D}[f] + \frac{\Omega^2}{\pi \rho_0} \left[ \frac{21}{20h} \mathcal{D}[f] + \frac{3}{4h^2} \mathcal{D}[h^2 p_{123}] \right] \right\}_{\lambda_1 = \lambda_2} \end{aligned} \quad (166)$$

is a function of the eccentricity. The last step consists in evaluating this condition along the equilibrium sequence. Using equations (146) and (148) we obtain

$$\frac{\Omega^2}{\pi\rho_0} = 2B_{11} + \frac{2M_c}{a_1}C(e), \quad (167)$$

where

$$C(e) = P(e) - \frac{1}{9}E(e), \quad (168)$$

and the leading term  $2B_{11}$  formally coincides with the Newtonian expression (30). The equation for  $C(e)$  has been derived by using MAPLE, using again the derivatives given in LRS [expressions (A3), (A9)] and the equations reported in Appendix D. The value of the eccentricity at the secular instability point,  $e_{sec}$ , is calculated by equating the two expressions (146) and (167) and is reported in Table 3. Figure 6 shows the ratio  $T/|W|$  evaluated at  $e = e_{sec}$ , as a function of the compactness parameter  $M/R_s$ . Squares in Figure 3 mark the instability as a function of the two gauge invariant ratios  $\Omega^2/(\pi\rho_0)$  and  $T/|W|$ , and thus separate the regions of stable and unstable configurations. As we can see, effect of general relativity is to move the instability point to an eccentricity larger than what Newtonian theory predicts. This corresponds also to larger values of the two ratios  $\Omega^2/(\pi\rho_0)$  and  $T/|W|$  (see Figure 3), showing, therefore, that relativistic gravitation tends to stabilize a star against secular instability to a Jacobi-like, nonaxisymmetric bar mode.

## 7. Discussion and Conclusions

In PN gravitation the rotational velocity along the equilibrium sequence is not much different from that one derived in the Newtonian limit, although a star of a given eccentricity has a slightly larger value of  $\Omega^2/(\pi\rho_0)$ . General relativity, however, is more crucial in influencing the stability properties. According to our treatment, the critical value of the eccentricity for the onset of the bar instability increases as the star becomes more relativistic, in the regime in which the PN approximation is valid. Both invariant ratios,  $\Omega^2/(\pi\rho_0)$  and  $T/|W|$ , also increase at the onset of instability above the values found in Newtonian theory. Gravitational radiation does not enter at PN order; it is present only at the 2.5 PN level and higher. The secular instability we have identified is driven by the presence of viscosity. That conclusion is consistent with our variational procedure, whereby the rotation was kept uniform and angular momentum was conserved. Viscous dissipation conserves angular momentum and drives a star to uniform rotation. Gravitational radiation dissipation conserves circulation (Miller 1974; LRS), not angular momentum, and does not maintain uniform rotation. The presence of a stabilizing effect due to general relativity on the Jacobi-like bar mode instability is in agreement with the BFG and Bonazzola, Friebe & Gourgoulhon (1997) analysis of relativistic polytropes, who observe a growth of the critical adiabatic index when the relativistic character of the configuration is increased. Shear viscosity can provide a such dissipation mechanism in a very cold NS ( $T \lesssim 10^6$  K), but it is inefficient for hot, newborn objects with  $T \sim 10^{10}$  K (see e.g. BFG), where the nonaxisymmetric instability is more probably induced by gravitational radiation (the CFS instability) and proceeds via a Dedekind-like mode. In this case, the fully relativistic SF computations show that effects of general relativity are reversed, and that the instability is significantly strengthened with respect to Newtonian theory. SF calculations are fully relativistic, so this bimodal behaviour may be explained in terms of strong field effects. However, as originally suggested by SF, there is also the likelihood that, in general relativity, the viscosity driven and the gravitational radiation driven  $m = 2$  modes may no longer coincide and that the gravitational field plays a different role in each of them. Although at present no firm conclusion can be reached, this speculation is further strengthened by the results of our PN treatment. Unfortunately, a

direct comparison between our results and the ones obtained by BFG and SF is not possible, since in both that cases numerical constraints prevented them from treating incompressible fluid configurations, such as the Maclaurin and Jacobi ellipsoids, where the density profile is strongly discontinuous at the surface. In the BFG’s case, numerical tests showed that the density steepens dramatically (and the numerical error increases more and more) for  $\gamma > 3.25$  ( $n < 0.44$ ). Models presented by SF are restricted to values of the polytropic index  $n \geq 1$ , since the presence of discontinuous derivatives of energy density and metric functions across the surface of the configuration make the description less accurate for stiffer equations of state. Moreover, the  $m = 2$  mode is only present in the SF model with  $n = 1$ .

In Newtonian theory, the variational principle also has been used by LRS to investigate the secular instability to dissipation which conserves circulation  $\mathcal{C}$ , rather than  $J$ , such as the emission of gravitational waves. In this case they used the energy functional for Dedekind ellipsoid, rather than a Jacobi ellipsoid as in the case of a viscosity-driven mode. Because the energy function of a Riemann-S ellipsoid is symmetric under interchange of  $\mathcal{C}$  and  $J$ , the two analysis are virtually identical in Newtonian theory, with  $\mathcal{C}$  appearing in place of  $J$  in all results. The previous discussion suggest that this symmetry may be broken in a relativistic treatment; this may be apparent at PN order and we plan to investigate this issue in a forthcoming work.

We are grateful to J.L. Friedman and N. Stergioulas for providing a copy of Stergioulas’s Ph.D Thesis prior to publication. We thank also Thomas Baumgarte for several very helpful discussions. This work was supported by NSF Grants AST 96–18524 and NASA Grant NAG 5–3420 at the University of Illinois at Urbana–Champaign.

## A. Comparison with the Chandrasekhar’s Results

Results presented here have been derived by using a 3+1 decomposition of the Einstein field equations. It is therefore useful to compare the resulting expressions with those obtained by CN, who worked with the usual field equations. In the first PN approximation, the metric coefficients  $g_{00}$ ,  $g_{0\alpha}$  and  $g_{\alpha\beta}$  are retained up to the order  $O(c^{-4})$ ,  $O(c^{-3})$  and  $O(c^{-2})$ , respectively (see CN and footnote 4 in this paper). In this Appendix we report the comparison. In section A.1 we show that the same gauge condition is common to both treatments. In section A.2 we compare the metric functions, while in A.3 we compare the conserved quantities.

### A.1. The Spatial Gauge Condition

Here we verify that the gauge condition adopted in CN is identical to our own. Chandrasekhar’s expressions are obtained in the gauge

$$\sum_i \frac{\partial P_i}{\partial x_i} = -3 \frac{\partial U}{\partial t} + O\left(\frac{1}{c^{-2}}\right), \quad (\text{A1})$$

where  $P_i = -\beta_{PN}^i$  [see CN, equations (3) and (6); section A.2.2 in this paper]. In our approach the condition is [see CST96, equation (3)]

$$\frac{\partial \ln \gamma^{1/2}}{\partial t} = D_k \beta^k, \quad (\text{A2})$$

where  $\gamma = \Psi^{12}$ . At leading order, we have the approximate expressions

$$D_k \beta^k \approx \sum_k \frac{\partial \beta_{PN}^k}{\partial x_k}, \quad \frac{\partial \ln \gamma^{1/2}}{\partial t} \approx \frac{6}{\gamma} \frac{\partial \Psi}{\partial t} \approx 6 \frac{\partial \Psi}{\partial t}. \quad (\text{A3})$$

Substituting  $\Psi = 1 - \Phi_N/2$  and neglecting terms of higher order, we obtain

$$-3 \frac{\partial \Phi_N}{\partial t} \approx \sum_k \frac{\partial \beta_{PN}^k}{\partial x_k}, \quad (\text{A4})$$

which coincides with the Chandrasekhar's choice (A1).

## A.2. The Metric

### A.2.1. The Lapse Function and the Conformal Factor

We can now compare expressions (63) and (64) for  $\alpha_N$  and  $\alpha_{PN}$  with the corresponding terms derived by CN. We start from the CN's expression of  $g_{00}$

$$g_{00} \approx 1 - 2U + 2(U^2 - 2\Phi_{Ch}), \quad (\text{A5})$$

where  $\Phi_{Ch}$  is the solution of

$$\nabla^2 \Phi_{Ch} = -4\pi\rho_0 \left( v^2 + U + \frac{3}{2} \frac{P}{\rho_0} \right) \quad (\text{A6})$$

and  $U = -\Phi_N$  [see CN, equations (3) and (4)]. Note that CN used a different signature for the metric, i.e. + - - -. At our order of approximation, we find

$$\begin{aligned} \alpha_{Ch} &\approx \sqrt{g_{00}} \approx 1 - U + U^2 - 2\Phi_{Ch} - \frac{U^2}{2} \\ &\approx 1 - U + \frac{U^2}{2} - 2\Phi_{Ch}. \end{aligned} \quad (\text{A7})$$

On the other hand,  $\Phi_{PN}$  and  $\Theta_{PN}$  are defined as the solutions of [equation (58), (61)]

$$\nabla^2 \Phi_{PN} = -10\pi\rho_0 \Phi_N + 4\pi\rho_0 v^2, \quad (\text{A8})$$

$$\nabla^2 \Theta_{PN} = 6\pi\rho_0 \left( v^2 + 2\frac{P}{\rho_0} - \frac{1}{2}\Phi_N \right), \quad (\text{A9})$$

so that, by comparing the corresponding differential equations, we can immediately recognize that  $-\Phi_{Ch} = \Phi_{PN}/4 + \Theta_{PN}/2$ . Substituting this result into (A7) gives

$$\alpha_{Ch} \approx 1 + \Phi_N + \frac{\Phi_N^2}{2} + \frac{\Phi_{PN}}{2} + \Theta_{PN}, \quad (\text{A10})$$

which agrees with the solution derived in section 4.1.

For the spatial metric the comparison is trivial. The PN expression in CN is [CN, equation (3)]

$$g_{ab} \approx -(1 + 2U) \delta_{ab}, \quad (\text{A11})$$

and, at the PN order,  $\Psi^4 = -g_{ab} \approx (1 + U/2) \delta_{ab}$ .

### A.2.2. The Shift Function

Consider now the shift vector. In the CN’s formalism, at the PN order it is [CN, equations (3), (5)]

$$g_{0a} \approx g^{0a} \approx P_a, \quad (\text{A12})$$

where  $P_a$  is the solution of the equation

$$\nabla^2 P_a = -16\pi\rho_0 v_a + \frac{\partial^2 U}{\partial x_a \partial t}. \quad (\text{A13})$$

Using the gauge condition [CN, equation (6)]

$$\frac{\partial U}{\partial t} \approx -\frac{1}{3} \sum_i \frac{\partial P_i}{\partial x_i}, \quad (\text{A14})$$

and neglecting terms of higher order, expression (A13) can be rewritten as

$$\nabla^2 P_a \approx -16\pi\rho_0 v_a - \frac{1}{3} \frac{\partial}{\partial x_a} \sum_i \frac{\partial P_i}{\partial x_i}. \quad (\text{A15})$$

On the other hand, in our formalism we have [equations (42), (44), (70), (71)]

$$\begin{aligned} \nabla^2 \beta^i &= \nabla^2 G^i - \frac{1}{4} \nabla^2 (\nabla^i B) \\ &\approx 16\pi\rho_0 v^i - \frac{1}{4} \nabla^i \nabla_k G^k. \end{aligned} \quad (\text{A16})$$

Taking the divergence of  $\beta^k$  and using equation (44) again gives

$$\nabla_k \beta^k = \nabla_k G^k - \frac{1}{4} \nabla^2 B = \frac{3}{4} \nabla_k G^k, \quad (\text{A17})$$

which can be substituted into (A16), yielding  $P_i = -\beta_{PN}^i$ .

### A.3. The Conserved Quantities

The derivation of conservation laws in general relativity has been considered by Chandrasekhar (1969b) and CN, who used the symmetric energy–momentum complex of Landau–Lifshitz to determine integral forms of conserved quantities at various post–Newtonian orders.<sup>6</sup> In this Appendix we compare our results (106) and (114) with the corresponding integrands obtained by CN. Consider the conserved energy. In the first PN approximation, CN derive the energy per unit volume of the fluid [see CN, equation (67)]

$$\begin{aligned} \mathcal{E} &= \rho_0 \left( \frac{1}{2} v^2 - \frac{1}{2} U + \Pi \right) \\ &+ \rho_0 \left[ \frac{5}{8} v^4 + \frac{5}{2} v^2 U - \frac{5}{2} U^2 + 2U\Pi + v^2 \left( \Pi + \frac{P}{\rho_0} \right) - \frac{1}{2} v_i P_i \right], \end{aligned} \quad (\text{A18})$$

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<sup>6</sup>Actually, in the first PN approximation the same expressions were originally obtained by Ch65a from a direct inspection of the PN generalization of the equations of motion and assuming the supplementary condition for isentropic flow. However, the fact that it is possible to derive the conserved quantities without the aid of the Landau–Lifshitz complex (at least when the additional condition is provided) is a peculiarity of the first PN order. The method discussed in Chandrasekhar (1969b) appears to be the most convenient way, allowing the derivation of the integral quantities in all the PN approximations beyond the first.

where  $\Pi$  is the internal energy and  $P_i = -(\beta_i)_{PN}$ . In the incompressible case of interest here,  $\Pi = 0$ . In order to compare (A18) with the integrand function appearing in our expression (106), we note that CN's results have been derived exploiting a set of identities between quantities that are equal, modulo a divergence. In fact, upon integration, two functions that are equal modulo divergence give the same conserved quantities (see Chandrasekhar 1969b for a detailed discussion). After some lengthy reductions, it is possible to demonstrate that the  $K^{ij}K_{ij}$  contribution appearing in (106) can be rewritten as

$$K^{ij}K_{ij} = \frac{1}{2} \left( \frac{\partial P_a}{\partial x_b} \frac{\partial P_a}{\partial x_b} + \frac{\partial P_a}{\partial x_b} \frac{\partial P_b}{\partial x_a} \right) - \frac{1}{3} \left( \text{div} \vec{P} \right)^2. \quad (\text{A19})$$

From equations (62), (63) of CN we have

$$\frac{\partial P_a}{\partial x_b} \frac{\partial P_a}{\partial x_b} \equiv 16\pi\rho_0 v_i P_i - 3 \left( \frac{\partial U}{\partial t} \right)^2 \pmod{\text{div}}, \quad (\text{A20})$$

$$\frac{\partial P_a}{\partial x_b} \frac{\partial P_b}{\partial x_a} \equiv 9 \left( \frac{\partial U}{\partial t} \right)^2 \pmod{\text{div}}, \quad (\text{A21})$$

and these equalities can be used, together with the gauge condition (A1)

$$-3 \frac{\partial U}{\partial t} = \text{div} \vec{P}, \quad (\text{A22})$$

to rearrange expression (A19). This yields

$$\begin{aligned} K^{ij}K_{ij} &\equiv 8\pi\rho_0 v_i P^i + \frac{1}{3} \left( \text{div} \vec{P} \right)^2 - \frac{1}{3} \left( \text{div} \vec{P} \right)^2 \pmod{\text{div}} \\ &\equiv -8\pi\rho_0 (v_i \beta^i)_{PN} \pmod{\text{div}}. \end{aligned} \quad (\text{A23})$$

Finally, inserting back (A23) into (106) one recognizes that our expression for the energy density agrees with equation (A18).

Consider next the angular momentum. Start from expression (141) by Ch65a for the conservation of total angular momentum,

$$\int_V (x_i \pi_j - x_j \pi_i) d^3x = \text{constant}, \quad (\text{A24})$$

where

$$\pi_i = \sigma v_i + \frac{1}{2} \rho_0 (U_i - U_{k;ik}) + 4\rho_0 (v_i U - U_i), \quad (\text{A25})$$

$U_i$ ,  $U_{k;ik}$  are quantities which enter the shift function,  $\sigma = \rho_0 (1 + v^2 + 2U + P/\rho_0)$ , and where incompressibility implies  $\Pi = 0$ . Specializing to our velocity profile, we have  $\pi_3 = 0$  and

$$J_{Ch} = \int_V (x_1 \pi_2 - x_2 \pi_1) d^3x. \quad (\text{A26})$$

After some algebra it is possible to demonstrate that the PN expression of the shift function in Ch65a is

$$(\beta_{PN}^i)_{CH} \approx -\alpha^2 g^{0i} \approx -g^{0i} = -4U_i + \frac{1}{2} (U_i - U_{k;ik}) = -\frac{7}{2} U_i - \frac{1}{2} U_{k;ik}, \quad (\text{A27})$$

and that the integrand appearing into (A26) can be rearranged as

$$\begin{aligned} x_1 \pi_2 - x_2 \pi_1 &= \frac{\rho_0}{\Omega} \left( v^2 + v^4 + 6Uv^2 + \frac{P}{\rho_0} v^2 \right) \\ &- \frac{7}{2} \rho_0 x_1 U_2 - \frac{1}{2} \rho_0 x_1 U_{k;2k} + \frac{7}{2} \rho_0 x_2 U_1 + \frac{1}{2} \rho_0 x_2 U_{k;1k} + \dots \end{aligned} \quad (\text{A28})$$

Rewriting the last row in the previous formula as

$$\rho_0 x_1 \beta^2 - \rho_0 x_2 \beta_1 = \frac{\rho_0}{\Omega} (v_2 \beta^2 + v_1 \beta^1) \quad (\text{A29})$$

and inserting (A28), (A29) into the conserved quantity (A26), we finally recover the same expression as in (114).

## B. The Spherical Limit

### B.1. Comparison with the Exact Solution

In the spherical limit, the interior metric for a homogeneous configuration admits an exact solution in full general relativity. In this Appendix we verify that our results agree with the PN expansion of the exact solution. We start from the interior 3-metric, written in Schwarzschild and in conformal (isotropic) coordinates as

$$ds_{(3)}^2 = \frac{1}{1 - 2m(r_s)/r_s} dr_s^2 + r_s^2 d\Omega^2 = \Psi^4 [dr^2 + r^2 d\Omega^2] , \quad (\text{B1})$$

where  $m(r_s) = 4\pi\rho_0 r_s^3/3$ . Equating the metric coefficients gives

$$\Psi^2 dr = \frac{1}{\sqrt{1 - 2m(r_s)/r_s}} dr_s \quad (\text{B2})$$

$$r_s = \Psi^2 r , \quad (\text{B3})$$

and these relations can be combined to yield the differential equation

$$\frac{2}{\Psi} d\Psi = \left( 1 - \frac{1}{\sqrt{1 - 2m(r_s)/r_s}} \right) \frac{dr_s}{r_s} . \quad (\text{B4})$$

Integrating (B4) gives

$$\Psi = K \left( 1 + \sqrt{1 - \frac{8\pi\rho_0}{3} r_s^2} \right)^{1/2} , \quad (\text{B5})$$

where  $K$  is a constant of integration. By using (B3), expression (B5) can be written in terms of isotropic coordinates as

$$\Psi = \frac{\sqrt{2}K}{\left( 1 + K^4 \frac{8\pi\rho_0}{3} r^2 \right)^{1/2}} . \quad (\text{B6})$$

The value of  $K$  is determined by matching the solution with the exterior metric. This is [see Misner, Thorne & Wheeler, equation (31.22), page 840]

$$\Psi = 1 + \frac{M}{2r} , \quad (\text{B7})$$

and, at the surface of the star,  $r = R$ . Imposing the boundary condition yields

$$\sqrt{2}K = \left( 1 + \frac{M}{2R} \right)^{3/2} . \quad (\text{B8})$$

The PN expression of  $\Psi$  can be then derived expanding (B6), and using the relations

$$M = \frac{4\pi\rho_0}{3} R_s^3 , \quad (\text{B9})$$



$$R_s = R \left( 1 + \frac{M}{2R} \right)^2. \quad (\text{B10})$$

This yields

$$\Psi \approx 1 + \pi \rho_0 R^2 - \frac{\pi \rho_0}{3} r^2 + \frac{25}{6} (\pi \rho_0)^2 R^4 - \frac{5}{3} (\pi \rho_0)^2 R^2 r^2 + \frac{1}{6} (\pi \rho_0)^2 r^4. \quad (\text{B11})$$

By specializing our solution (73) and (76) to the spherical case gives

$$\Phi = \Phi_N + \Phi_{PN} = -2\pi \rho_0 R^2 + \frac{2\pi \rho_0}{3} r^2 - \frac{25}{3} (\pi \rho_0)^2 R^4 + \frac{10}{3} (\pi \rho_0)^2 R^2 r^2 - \frac{1}{3} (\pi \rho_0)^2 r^4. \quad (\text{B12})$$

which, with (59), agrees with (B11).

## B.2. The Relativistic Correction to the Total Energy

Our integrated expression for  $M - M_0$  allows a comparison with the relativistic correction to the total energy of a spherical configuration as derived by ST (see also Zel'dovich & Novikov, 1971). Following ST, let us define the relativistic correction as

$$\Delta E_{GTR} \equiv M - M_0 - E_N, \quad (\text{B13})$$

where  $E_N$  is the Newtonian gravitational energy

$$E_N = -\frac{3}{5} \frac{M_N^2}{R_N} = -\frac{3}{5} \left( \frac{4\pi}{3} \right)^2 \rho_0^2 R_N^5, \quad (\text{B14})$$

and  $M_N$ ,  $R_N$  are the Newtonian mass and radius, respectively. Dimensionally, at PN order this correction can be written as

$$\Delta E_{GTR} = -k M^{7/3} \rho_0^{2/3}, \quad (\text{B15})$$

and the value of  $k$  in the homogeneous case can be obtained by using equation (6.9.29) in ST for an incompressible gas with polytropic index  $n = 0$ . This yields<sup>7</sup>

$$k = \frac{3}{70} \left( \frac{4\pi}{3} \right)^{2/3}. \quad (\text{B16})$$

In order to recover this result from our equations, we specialize equation (127) to the nonrotating, spherical case. In this limit  $A_1 = A_2 = A_3 = 2/3$  and  $f = 1$ , so that

$$\begin{aligned} M - M_0 &\approx -\frac{3}{5} \frac{M_c^2}{R} - \frac{51}{14} \frac{M_c^3}{R^2} \\ &= -\frac{3}{5} \left( \frac{4\pi}{3} \right)^2 \rho_0^2 R^5 - \frac{51}{14} \left( \frac{4\pi}{3} \right)^3 \rho_0^3 R^7. \end{aligned} \quad (\text{B17})$$

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<sup>7</sup>Note that, in the limit  $n \rightarrow 0$ , the two integrals appearing in ST (6.9.29) become  $I_1 = -3M^3/(14R_N^2)$  and  $I_2 = -3M^3/(35R_N^2)$ . This gives

$$\Delta E_{GTR} = -\frac{3}{70} \frac{M^3}{R_N^2} = -\frac{3}{70} \left( \frac{4\pi}{3} \right)^{2/3} M^{7/3} \rho_0^{2/3}.$$

On the other hand, the exact relation between the conformal (isotropic) radial coordinate  $R$  and the Schwarzschild radius  $R_s$  is given by (B10). At PN order

$$R \approx R_s \left( 1 - \frac{M}{R_s} \right), \quad (\text{B18})$$

and we left to derive the relation between  $R_s$  and  $R_N$ . Start from the definition (6.9.12) in ST

$$R_N \equiv \left( \frac{3V}{4\pi} \right)^{1/3}, \quad (\text{B19})$$

where  $V$  is the proper volume. To first order [see (6.9.15)]

$$\begin{aligned} R_N &\approx R_s \left( 1 + \frac{1}{R_s^3} \int_0^{R_s} m r_s dr_s \right) \\ &\approx R_s \left( 1 + \frac{1}{R_s} \frac{4\pi\rho_0}{3} \int_0^{R_s} r_s^4 dr_s \right) \approx R_s \left( 1 + \frac{M}{5R_s} \right). \end{aligned} \quad (\text{B20})$$

This gives

$$R_s \approx R_N \left( 1 - \frac{M}{5R_N} \right), \quad (\text{B21})$$

and, combining (B18) and (B21), yields

$$R \approx R_N \left( 1 - \frac{6}{5} \frac{M}{R_N} \right). \quad (\text{B22})$$

Finally, substituting the equation (B22) into equation (B17) and subtracting  $E_N$ , we obtain the relativistic correction

$$M - M_0 - E_N \approx -\frac{3}{70} \frac{M_N^3}{R_N^2} \approx -\frac{3}{70} \frac{M^3}{R_N^2}, \quad (\text{B23})$$

which agrees with the ST's result.

### C. The Integral $\mathcal{I}_{10}$

The integral  $\mathcal{I}_{10}$  has been obtained by using MAPLE. We only report here the resulting expression, that is

$$\begin{aligned} \tilde{\mathcal{I}}_{10} &= (39/560)(A_1 - A_2)^2 / (\lambda_1 \lambda_2) - (9/140) \bar{A}_{12}^2 (1 - \lambda_1^3 / \lambda_2^3) (\bar{A}_{122} - \bar{A}_{112} \lambda_1^3 / \lambda_2^3) \lambda_1^2 / \lambda_2^4 \\ &+ (3/2240)(A_1 - A_2)^2 (9 \bar{A}_{112}^2 + \bar{A}_{122}^2 + (-2A_2 \lambda_2^3 + 2\lambda_2^3 - A_1 \lambda_2^3) \\ &- \bar{A}_{12} \lambda_1^3 + \bar{A}_{12} \lambda_1^3 \lambda_2^3)^2 / (1 - \lambda_2^3)^2 / \lambda_1^6 / (1 - \lambda_1^3)^2 \lambda_1^5 / \lambda_2^7 \\ &+ (3/8)(\lambda_1^2 / \lambda_2^4) (-1/5(A_1 - A_2 \lambda_1^3 / \lambda_2^3)^2 + (3/35) \lambda_1^6 (-1/(1 - \lambda_1^3)(2A_1 - 2 + A_2) \\ &+ 1/(1 - \lambda_2^3)(2A_2 - 2 + A_1))^2 + (6/35) \lambda_1^3 (-1/(1 - \lambda_1^3)(2A_1 - 2 + A_2) + 1/(1 - \lambda_2^3) \\ &\times (2A_2 - 2 + A_1))(A_1 - (1/3) \bar{A}_{12} (\lambda_1^3 / \lambda_2^3) + 1/3/(1 - \lambda_1^3)(2A_1 - 2 + A_2) \lambda_1^3 \\ &- (\lambda_1^3 / \lambda_2^3)(A_2 - (1/3) \bar{A}_{12} + 1/3/(1 - \lambda_2^3)(2A_2 - 2 + A_1) \lambda_2^3))) + (3/280) \bar{A}_{12} (1 \\ &- \lambda_1^3 / \lambda_2^3) (\lambda_1^2 / \lambda_2^4) (3 \bar{A}_{12} (A_1 - A_2 \lambda_1^3 / \lambda_2^3) - 9A_1 \bar{A}_{112} \\ &+ 9A_2 (\lambda_1^6 / \lambda_2^6) \bar{A}_{122} + 3(\bar{A}_{112} - \bar{A}_{122} \lambda_1^3 / \lambda_2^3) \bar{A}_{12} \lambda_1^3 / \lambda_2^3 \end{aligned} \quad (\text{C1})$$

$$\begin{aligned}
& - \lambda_1^3/(1 - \lambda_1^3)(2A_1 - 2 + A_2)(3\bar{A}_{112} - (-2A_2\lambda_2^3 + 2\lambda_2^3 - A_1\lambda_2^3 - \bar{A}_{12}\lambda_1^3 \\
& + \bar{A}_{12}\lambda_1^3\lambda_2^3)/(-1 + \lambda_2^3)/(-1 + \lambda_1^3)) + \lambda_1^3/(1 - \lambda_2^3)(2A_2 - 2 + A_1)(3\bar{A}_{122}\lambda_1^3/\lambda_2^3 \\
& - (-2A_2\lambda_2^3 + 2\lambda_2^3 - A_1\lambda_2^3 - \bar{A}_{12}\lambda_1^3 + \bar{A}_{12}\lambda_1^3\lambda_2^3)/(-1 + \lambda_2^3)/(-1 + \lambda_1^3))) \\
& + (3/70)(\lambda_2^2/\lambda_1)(1/(1 - \lambda_2^3)^2(2A_2 - 2 + A_1)^2\lambda_1^3/\lambda_2^3 + 1/(1 - \lambda_1^3)^2(2A_1 - 2 + A_2)^2) \\
& + (3/1120)\bar{A}_{12}^2(1 - \lambda_1^3/\lambda_2^3)^2(1 + \lambda_1^3/\lambda_2^3)(-2A_2\lambda_2^3 + 2\lambda_2^3 - A_1\lambda_2^3 - \bar{A}_{12}\lambda_1^3 \\
& + \bar{A}_{12}\lambda_1^3\lambda_2^3)^2/(-1 + \lambda_2^3)^2/\lambda_1/(-1 + \lambda_1^3)^2/\lambda_2^4 - (3/140)\bar{A}_{12}(1 \\
& - \lambda_1^3/\lambda_2^3)(-1/(1 - \lambda_1^3)(2A_1 - 2 + A_2) + \lambda_1^3/\lambda_2^3/(1 - \lambda_2^3)(2A_2 - 2 + A_1))(-2A_2 * \lambda_2^3 \\
& + 2\lambda_2^3 - A_1\lambda_2^3 - \bar{A}_{12}\lambda_1^3 + \bar{A}_{12}\lambda_1^3\lambda_2^3)/(-1 + \lambda_2^3)/\lambda_1/(-1 + \lambda_1^3)/\lambda_2 \\
& + (3/128)\bar{A}_{12}^2(1 - \lambda_1^3/\lambda_2^3)^2(\lambda_1^2/\lambda_2^4)(-\bar{A}_{12}^2 + (6/7)\bar{A}_{12}\bar{A}_{112} + (9/35)\bar{A}_{112}^2 \\
& + (2/7)\bar{A}_{12}(-2A_2\lambda_2^3 + 2\lambda_2^3 - A_1\lambda_2^3 - \bar{A}_{12}\lambda_1^3 \\
& + \bar{A}_{12}\lambda_1^3\lambda_2^3)/(-1 + \lambda_2^3)/(-1 + \lambda_1^3) + (1/35)(-2A_2\lambda_2^3 + 2\lambda_2^3 - A_1\lambda_2^3 - \bar{A}_{12}\lambda_1^3 \\
& + \bar{A}_{12}\lambda_1^3\lambda_2^3)^2/(-1 + \lambda_2^3)^2/(-1 + \lambda_1^3)^2 + (27/35)\bar{A}_{122}^2\lambda_1^6/\lambda_2^6 - (6/35)\bar{A}_{122}(-2A_2\lambda_2^3 \\
& + 2\lambda_2^3 - A_1\lambda_2^3 - \bar{A}_{12}\lambda_1^3 + \bar{A}_{12}\lambda_1^3\lambda_2^3)/(-1 + \lambda_2^3)\lambda_1^3/(-1 \\
& + \lambda_1^3)/\lambda_2^3) + (3/10)\bar{A}_{12}^2\lambda_1^2/\lambda_2^4 + (81/280)(\lambda_2^2/\lambda_1^4)(A_1 - (1/3)\bar{A}_{12}\lambda_1^3/\lambda_2^3 + 1/3/(1 - \lambda_1^3)(2A_1 \\
& - 2 + A_2)\lambda_1^3)^2 + (81/280)(\lambda_2^2/\lambda_1^4)(A_2 - (1/3)\bar{A}_{12} + 1/3/(1 - \lambda_2^3)(2A_2 - 2 + A_1)\lambda_2^3)^2 \\
& - (27/140)(A_1 - (1/3)\bar{A}_{12}\lambda_1^3/\lambda_2^3 + 1/3/(1 - \lambda_1^3)(2A_1 - 2 + A_2)\lambda_1^3)(A_2 - (1/3)\bar{A}_{12} + 1/3 \\
& - 3\lambda_2^3)(2A_2 - 2 + A_1)\lambda_2^3/\lambda_1/\lambda_2 - (9/70)\bar{A}_{12}(A_1 + A_2\lambda_1^3/\lambda_2^3)/\lambda_1/\lambda_2 + (3/70)\bar{A}_{12}\lambda_1^2 \\
& \times (-1/(1 - \lambda_1^3)(2A_1 - 2 + A_2) - 1/(1 - \lambda_2^3)(2A_2 - 2 + A_1))/\lambda_2,
\end{aligned}$$

where

$$\bar{A}_{12} = a_1^2 A_{12}, \quad \bar{A}_{122} = a_1^4 A_{122}, \quad \bar{A}_{112} = a_1^4 A_{112}, \quad (C2)$$

only depend on the axial ratios.

#### D. The Index Symbols

The expressions of the metric functions we have derived have been presented in terms of the index symbols  $A_{ijk\dots}$  and  $B_{ijk\dots}$ , where

$$A_{ijk\dots} = a_1 a_2 a_3 \int_0^\infty \frac{du}{\Delta(a_i^2 + u)(a_j^2 + u)(a_k^2 + u)\dots}, \quad (D1)$$

$$B_{ijk\dots} = a_1 a_2 a_3 \int_0^\infty \frac{u du}{\Delta(a_i^2 + u)(a_j^2 + u)(a_k^2 + u)\dots}, \quad (D2)$$

and where  $\Delta^2 = (a_1^2 + u)(a_2^2 + u)(a_3^2 + u)$ . The use of these quantities is particularly convenient, since it allows for a more compact form of the results. In the expressions of the conserved quantities, however, it is more convenient to reduce the number of different index symbols entering in the final forms. This is because when we perform the variations of the conserved quantities by using MAPLE, the derivatives of the index symbols are evaluated analytically and then substituted in the variations. The index symbols are not independent: they are manifestly symmetric in their indices and obey to a set of identities and recursive relations (see Chapter 3 in Ch69). Exploiting these properties we derived the results presented in

D.1; these have been used to obtain the final expressions of the  $g_i$  and  $p_i$  functions which enter in  $M$ ,  $M_0$  and  $J$  [see (128)–(134), (C1)]. Note that the final forms only depends on the quantities

$$A_i, \bar{A}_{12}, \bar{A}_{122}, \bar{A}_{112}. \quad (\text{D3})$$

Then, variations have been performed by using a MAPLE program. The final results have been specialized to axisymmetry, using expressions (D13), (D14) and substituting the derivatives given in D.2.

### D.1. The Coefficients $A_{ijk}$ and $B_{ijk}$ ; General Expressions

The  $A_i$ 's functions are given in terms of the standard (elliptic) integrals by expressions (3.33)–(3.35) in Ch69. The other index symbols can be written as

$$A_{il} = -\frac{1}{a_i^2 - a_l^2} (A_i - A_l) \quad \text{for } i \neq l \quad (\text{D4})$$

$$\begin{aligned} A_{ii} &= \frac{2}{3a_i^2} - \frac{1}{3}A_{il} - \frac{1}{3}A_{im} && \text{for } i \neq l \neq m \\ &= \frac{2}{3a_i^2} - \frac{1}{3} \sum_j (1 - \delta_{ij}) A_{ij} && \end{aligned} \quad (\text{D5})$$

$$A_{ilm} = -\frac{1}{a_i^2 - a_l^2} (A_{im} - A_{lm}) \quad \text{for } i \neq l \quad (\text{D6})$$

$$A_{ill} = -\frac{1}{a_i^2 - a_l^2} (A_{il} - A_{ll}) \quad \text{for } i \neq l \quad (\text{D7})$$

$$\begin{aligned} A_{iil} &= \frac{5}{3a_i^2} A_{il} - \frac{a_l^2}{a_i^2} A_{ill} - \frac{A_{ilm}}{3} \frac{a_m^2}{a_i^2} && \text{for } i \neq l \neq m \\ &= \frac{5}{3a_i^2} A_{il} - \frac{1}{3a_i^2} \sum_m A_{ilm} a_m^2 (1 - \delta_{mi}) (1 + 2\delta_{lm}) && \text{for } i \neq l \end{aligned} \quad (\text{D8})$$

$$\begin{aligned} A_{iii} &= \frac{1}{a_i^2} A_{ii} - \frac{a_l^2}{5a_i^2} A_{iil} - \frac{a_m^2}{5a_i^2} A_{iim} && \text{for } i \neq l \neq m \\ &= \frac{1}{a_i^2} A_{ii} - \frac{1}{5a_i^2} \sum_m A_{iim} a_m^2 (1 - \delta_{im}) . && \end{aligned} \quad (\text{D9})$$

The  $B_{ilm..}$ 's are then:

$$B_i = I - a_i^2 A_i, \quad (\text{D10})$$

$$B_{ilm..} = A_{lm..} - a_i^2 A_{ilm..} . \quad (\text{D11})$$

By using these relations we obtain the final forms of  $p_i$ ,  $q_i$  and  $\mathcal{I}_{10}$ , which only contain the quantities

$$A_i, \bar{A}_{12}, \bar{A}_{122}, \bar{A}_{112}. \quad (\text{D12})$$

Moreover, the  $A_i$ 's are not independent, since  $A_3 = 2 - A_1 - A_2$ . The derivatives of these quantities have been evaluated analytically and are reported in the next section. In axisymmetry, we have  $A_3 = 2 - 2A_1$

and

$$\bar{A}_{12} = \frac{1}{2} + \frac{1}{4e^2} (3A_1 - 2), \quad (D13)$$

$$\bar{A}_{112} = \bar{A}_{122} = \frac{5}{6} \left[ \frac{1}{2} \left( 1 + \frac{1-e^2}{5e^2} \right) + \frac{1}{4e^4} (3A_1 - 2) \right]. \quad (D14)$$

## D.2. Some Useful Derivatives

The first and second derivatives of the quantities  $A_i$ ,  $\bar{A}_{12}$ ,  $\bar{A}_{122}$ ,  $\bar{A}_{112}$  with respect to the axial ratios have been evaluated using the chain rule

$$\frac{\partial A_{ijl}}{\partial \lambda_i} = \sum_k \frac{\partial A_{ijl..}}{\partial a_k} \frac{\partial a_k}{\partial \lambda_i}, \quad (D15)$$

and exploiting the definitions (D1). The resulting expressions have been evaluated in the axisymmetric case, and substituted into the variations of the conserved quantities by using MAPLE. The final results, already specialized to axisymmetry, are

$$\begin{aligned} \frac{\partial A_1}{\partial \lambda_1} &= \frac{15 A_1 + 12 A_1 \lambda^3 - 18 \lambda^3}{(8 - 8 \lambda^3) \lambda} \\ \frac{\partial A_1}{\partial \lambda_2} &= \frac{-3 A_1 + 12 A_1 \lambda^3 - 6 \lambda^3}{(8 - 8 \lambda^3) \lambda} \\ \frac{\partial A_2}{\partial \lambda_1} &= \frac{-3 A_1 + 12 A_1 \lambda^3 - 6 \lambda^3}{(8 - 8 \lambda^3) \lambda} \\ \frac{\partial A_2}{\partial \lambda_2} &= \frac{15 A_1 + 12 A_1 \lambda^3 - 18 \lambda^3}{(8 - 8 \lambda^3) \lambda} \\ \frac{\partial \bar{A}_{12}}{\partial \lambda_1} &= -\frac{9 A_1 + 18 \lambda^3 + 12 \lambda^6 - 54 A_1 \lambda^3}{16 (1 - \lambda^3)^2 \lambda} \\ \frac{\partial \bar{A}_{12}}{\partial \lambda_2} &= \frac{27 A_1 - 42 \lambda^3 + 12 \lambda^6 + 18 A_1 \lambda^3}{16 (1 - \lambda^3)^2 \lambda} \\ \frac{\partial \bar{A}_{122}}{\partial \lambda_1} &= \frac{-615 A_1 + 306 \lambda^3 - 1336 \lambda^6 + 400 \lambda^9 + 1560 A_1 \lambda^3}{384 (1 - \lambda^3)^3 \lambda} \\ \frac{\partial \bar{A}_{122}}{\partial \lambda_2} &= -\frac{-1095 A_1 + 2034 \lambda^3 - 1448 \lambda^6 + 464 \lambda^9 - 480 A_1 \lambda^3}{384 (1 - \lambda^3)^3 \lambda} \\ \frac{\partial \bar{A}_{112}}{\partial \lambda_1} &= \frac{15 A_1 - 738 \lambda^3 - 424 \lambda^6 + 112 \lambda^9 + 1560 A_1 \lambda^3}{384 (1 - \lambda^3)^3 \lambda} \\ \frac{\partial \bar{A}_{112}}{\partial \lambda_2} &= -\frac{-465 A_1 + 990 \lambda^3 - 536 \lambda^6 + 176 \lambda^9 - 480 A_1 \lambda^3}{384 (1 - \lambda^3)^3 \lambda}, \end{aligned} \quad (D16)$$

and

$$\frac{\partial^2 A_1}{\partial \lambda_1^2} = \frac{39 A_1 - 306 \lambda^3 + 516 A_1 \lambda^3 + 120 A_1 \lambda^6 - 144 \lambda^6}{32 (1 - \lambda^3)^2 \lambda^2}$$

$$\begin{aligned}
\frac{\partial^2 A_1}{\partial \lambda_1 \partial \lambda_2} &= \frac{-9 A_1 + 72 A_1 \lambda^3 - 18 \lambda^3 - 72 \lambda^6 + 72 A_1(\lambda, l_2) \lambda^6}{32 (1 - \lambda^3)^2 \lambda^2} \\
\frac{\partial^2 A_2}{\partial \lambda_1^2} &= \frac{3 A_1 + 6 \lambda^3 - 96 \lambda^6 + 12 A_1 \lambda^3 + 120 A_1 \lambda^6}{32 (1 - \lambda^3)^2 \lambda^2} \\
\frac{\partial^2 A_2}{\partial \lambda_1 \partial \lambda_2} &= \frac{-9 A_1 + 72 A_1 \lambda^3 - 18 \lambda^3 - 72 \lambda^6 + 72 A_1(\lambda, l_2) \lambda^6}{32 (1 - \lambda^3)^2 \lambda^2} \\
\frac{\partial^2 \overline{A}_{12}}{\partial \lambda_1^2} &= -\frac{-117 A_1 + 144 A_1 \lambda^3 - 234 \lambda^3 + 3048 \lambda^6 + 336 \lambda^9 - 4752 A_1 \lambda^6}{256 (1 - \lambda^3)^3 \lambda^2} \\
\frac{\partial^2 \overline{A}_{12}}{\partial \lambda_1 \partial \lambda_2} &= -\frac{-195 A_1 - 960 A_1 \lambda^3 + 634 \lambda^3 + 1144 \lambda^6 + 112 \lambda^9 - 1680 A_1 \lambda^6}{256 (1 - \lambda^3)^3 \lambda^2} \\
\frac{\partial^2 \overline{A}_{122}}{\partial \lambda_1^2} &= \frac{8325 A_1 - 20220 A_1 \lambda^3 - 7926 \lambda^3 + 18624 \lambda^6 - 21296 \lambda^9 + 4928 \lambda^{12} + 20400 A_1 \lambda^6}{512 \lambda^2 (1 - \lambda^3)^4} \\
\frac{\partial^2 \overline{A}_{122}}{\partial \lambda_1 \partial \lambda_2} &= -\frac{5535 A_1 - 10440 A_1 \lambda^3 - 7362 \lambda^3 + 21528 \lambda^6 - 11184 \lambda^9 + 2688 \lambda^{12} - 3600 A_1 \lambda^6}{512 \lambda^2 (1 - \lambda^3)^4} \\
\frac{\partial^2 \overline{A}_{112}}{\partial \lambda_1^2} &= \frac{-75 A_1 - 480 A_1 \lambda^3 + 1386 \lambda^3 - 12424 \lambda^6 - 2704 \lambda^9 + 20400 A_1 \lambda^6 + 512 \lambda^{12}}{512 \lambda^2 (1 - \lambda^3)^4} \\
\frac{\partial^2 \overline{A}_{112}}{\partial \lambda_1 \partial \lambda_2} &= -\frac{-345 A_1 - 4560 A_1 \lambda^3 + 2382 \lambda^3 + 3272 \lambda^6 + 16 \lambda^9 - 3600 A_1 \lambda^6}{512 \lambda^2 (1 - \lambda^3)^4}
\end{aligned} \tag{D17}$$

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Fig. 1.— The ratio  $\Omega^2/(\pi\rho_0)$  versus the eccentricity for PN equilibrium sequences of constant rest mass (solid lines). The different curves correspond to 11 equally spaced values of the compaction parameter  $M/R_s$  in the range  $[0., 0.275]$ . This parameter characterizes the nonrotating spherical member of each sequence. The dashed line is the Newtonian Maclaurin sequence ( $M/R_s = 0$ ).

Fig. 2.— The ratio  $T/|W|$  versus the eccentricity for the PN equilibrium sequence with  $(M/R_s)_{max} = 5/18$  (solid line), and the Newtonian Maclaurin sequence (dashed line).

Fig. 3.— The ratio  $\Omega^2/(\pi\rho_0)$  versus  $T/|W|$  for the same sequences as in Figure 1 (solid lines); the dashed line is the Newtonian Maclaurin sequence. Squares mark the secular instability point. .

Fig. 4.— Comparison between the ellipsoidal PN function  $E(e)$  (solid line) and the corresponding function  $E_{Ch}$  derived by Chandrasekhar (1965b, dotted line).

Fig. 5.— The ellipsoidal PN equilibrium sequence (solid line) and the Chandrasekhar (1965b) sequence (dotted line) for  $2M_c/a_1 = 0.206$ . Squares represent the Butterworth & Ipser (1976) numerical values, and the dashed line is the Newtonian Maclaurin sequence.

Fig. 6.— The critical ratio  $T/|W|$  at the secular instability point versus the compaction parameter  $M/R_s$  along a PN equilibrium sequence (solid line). The dashed line is the Newtonian value.

Table 1. The Function  $E(e)$ ; Comparison with Chandrasekhar (1965b)

$e$	$E(e)$	$E_{Ch}(e)$	err <sup>a</sup>		$e$	$E(e)$	$E_{Ch}(e)$	err <sup>a</sup>
0	0	0	–		0.65	0.1191	0.1202	0.0094
0.20	0.0090	0.0093	0.0397		0.70	0.1448	0.1444	0.0026
0.25	0.0142	0.0147	0.0389		0.75	0.1752	0.1719	0.0195
0.30	0.0207	0.0215	0.0379		0.80	0.2118	0.2026	0.0447
0.35	0.0287	0.0298	0.0366		0.8127*	0.2223	0.2108	0.0531
0.40	0.0383	0.0397	0.0350		0.85	0.2564	0.2355	0.0847
0.45	0.0480	0.0513	0.0661		0.90	0.3115	0.2666	0.1552
0.50	0.0629	0.0650	0.0268		0.9529**	0.3838	0.2768	0.3240
0.55	0.0790	0.0808	0.0220		0.96	0.3939	0.2723	0.3651
0.60	0.0974	0.0991	0.0177					

<sup>a</sup>The fractional difference is defined as

$$\text{err} \equiv 2 \frac{|E(e) - E_{Ch}(e)|}{E(e) + E_{Ch}(e)}.$$

\*Newtonian value for the secular instability point.

\*\*Newtonian value for the dynamical instability point.

Table 2. Comparison with Butterworth & Ipser (1976);  $2M_c/a_1 = 0.206^a$

$e$	$e_{BI}$	$\frac{\Omega^2}{\pi\rho_0}$	$\left(\frac{\Omega^2}{\pi\rho_0}\right)_{BI}$	$\frac{\Omega^2}{\pi\rho_0} - \left(\frac{\Omega^2}{\pi\rho_0}\right)_N$	$\left[\frac{\Omega^2}{\pi\rho_0} - \left(\frac{\Omega^2}{\pi\rho_0}\right)_N\right]_{BI}$	err <sup>b</sup>
0.199	0.200	0.024	0.025	0.002	–	–
0.298	0.300	0.054	0.058	0.006	–	–
0.397	0.400	0.097	0.107	0.011	–	–
0.498	0.500	0.155	0.164	0.018	–	–
0.598	0.600	0.227	0.247	0.027	–	–
0.695	0.700	0.314	0.329	0.040	0.051	0.24
0.796	0.800	0.419	0.429	0.059	0.066	0.10
0.846	0.850	0.472	0.474	0.072	0.066	0.08

<sup>a</sup>Corresponding to  $\gamma_s = 0.154$ , where  $\gamma_s$  is the parameter used by Butterworth & Ipser (1976).

<sup>b</sup>The fractional difference with respect the Butterworth & Ipser value is defined as

$$\text{err} \equiv 2 \frac{|Q - Q_{BI}|}{Q + Q_{BI}},$$

where  $Q = \Omega^2 / (\pi\rho_0) - [\Omega^2 / (\pi\rho_0)]_N$ .

Table 3. Location of the PN secular instability point

$\frac{M}{R_s}$	$e_{sec}$	$\left(\frac{\Omega^2}{\pi\rho_0}\right)_{sec}$	$\left(\frac{T}{ W }\right)_{sec}$
0. <sup>a</sup>	0.8127	0.3742	0.1375
0.025	0.8561	0.4230	0.1673
0.050	0.8850	0.4582	0.1924
0.075	0.9042	0.4831	0.2120
0.100	0.9171	0.5011	0.2268
0.125	0.9261	0.5144	0.2378
0.150	0.9325	0.5247	0.2458
0.175	0.9372	0.5327	0.2516
0.200	0.9403	0.5384	0.2554
0.225	0.9425	0.5424	0.2578
0.250	0.9436	0.5446	0.2591
0.275	0.9441	0.5457	0.2597

<sup>a</sup>Newtonian limit.













